NORMS OF EIGENFUNCTIONS TO TRIGONOMETRIC KZB OPERATORS

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Abstract. Let $\mathfrak{g}$ be a simple Lie algebra and $V[0] = V_1 \otimes \cdots \otimes V_n[0]$ the zero weight subspace of a tensor product of $\mathfrak{g}$-modules. The trigonometric KZB operators are commuting differential operators acting on $V[0]$-valued functions on the Cartan subalgebra of $\mathfrak{g}$. Meromorphic eigenfunctions to the operators are constructed by the Bethe ansatz. We introduce a scalar product on a suitable space of functions such that the operators become symmetric, and the square of the norm of a Bethe eigenfunction equals the Hessian of the master function at the corresponding critical point.

1. Introduction

We study three systems of commuting linear operators associated to a simple Lie algebra $\mathfrak{g}$ and the tensor product $V = V_1 \otimes \cdots \otimes V_n$ of finite dimensional representations of $\mathfrak{g}$. The first system is the collection of the rational Gaudin operators acting on the space of singular vectors of $M_{\xi-\rho} \otimes V$ of weight $\xi - \rho$, where $M_{\xi-\rho}$ is the Verma module of highest weight $\xi - \rho$. The second system is the collection of the trigonometric Gaudin operators with parameter ξ acting on the zero weight subspace $V[0] \subset V$. The third system is the collection the trigonometric KZB differential operators acting on a particular space $E(\xi)$ of $V[0]$-valued functions on the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The three systems are isomorphic.

Each system has a Bethe ansatz construction of eigenvectors. The Bethe ansatz consists of a scalar master function of auxiliary variables $t = (t_1, \dots, t_k)$ and a weight function depending on t. For a critical point t_{cr} of the master function, the value of the weight function at t_{cr} is an eigenvector.

A large body of the previous work focuses on the norms of such Bethe eigenvectors in the case of the rational Gaudin operators with respect to the Shapovalov form [MV], [V3]. In that case, if t_{cr} is an isolated critical point of the master function, then the norm of the eigenvector corresponding to t_{cr} equals the Hessian of the master function at t_{cr}. In this paper, we extend this result to the eigenvectors of the other two systems, see Section 6.

In Section 7, we describe the Weyl group action on eigenfunctions to the trigonometric KZB operators, and show that the scalar product on these functions is Weyl invariant. In section 8, we recall the construction of Jack polynomials from antisymmetrized eigenfunctions to the trigonometric KZB operators. We apply our results to relate the usual norm of Jack polynomials to the Hessian of the master function.

An interesting next step would be to establish similar results for the elliptic KZB operators.

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2. Preliminaries

2.1. **Notation.** Let $\mathfrak{g}$ be a simple complex Lie algebra of rank r with Cartan subalgebra $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots, and for $\alpha \in \Delta$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ denote the root space corresponding to α.

Fix simple roots $\alpha_1, \ldots, \alpha_r \in \Delta$. Let $Q = \oplus_j \mathbb{Z}\alpha_j$ be the root lattice, and Q_+ the elements of Q with non-negative coefficients. Let $\Delta_+ = \Delta \cap Q_+$ be the set of positive roots, and $\Delta_- = \Delta \setminus \Delta_+$, its complement, the negative roots. Let $\mathfrak{n}_\pm = \oplus_{\alpha\in\Delta_\pm} \mathfrak{g}_\alpha$ denote the positive and negative root spaces.

Fix a nondegenerate $\mathfrak{g}$-invariant bilinear form $(\, , \,)$ on $\mathfrak{g}$. The form identifies $\mathfrak{g}$ and $\mathfrak{g}^*$ and defines a bilinear form on $\mathfrak{g}^*$. For a root $\alpha \in \Delta$, we denote its coroot by $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, and set $h_\alpha \in \mathfrak{h}$ so that $(h_\alpha, h) = \alpha^\vee(h)$ for all $h \in \mathfrak{h}$.

For each $\alpha \in \Delta$, choose generators $e_\alpha \in \mathfrak{g}_\alpha$ so that $(e_\alpha, e_{-\alpha}) = 1$. For positive roots $\alpha \in \Delta_+$, we set $f_\alpha = \frac{2}{(\alpha_j,\alpha_j)} e_{-\alpha}$. For a simple root α_j, set $e_j = e_{\alpha_j}$, $f_j = f_{\alpha_j}$ and $h_j = h_{\alpha_j}$. Then $h_1, \ldots, h_r, e_1, \ldots, e_r, f_1, \ldots, f_r$ are Chevalley generators of $\mathfrak{g}$. Set $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha$, so $\rho(h_j) = 1$ for $j = 1, \ldots, r$.

Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. The Chevalley involution ω is an automorphism of $U(\mathfrak{g})$ defined by $\omega(e_j) = -f_j,\ \omega(f_j) = -e_j,\ \omega(h_j) = -h_j$. The antipode a is an anti-automorphism of $U(\mathfrak{g})$ defined by $a(g) = -g$ for $g \in \mathfrak{g}$.

2.2. **Shapovalov form.** The Poincaré-Birkhoff-Witt theorem gives the decomposition

$$U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+).$$

Denote by γ the projection $U(\mathfrak{g}) \to U(\mathfrak{h})$ along $\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+$.

The Shapovalov form is the $U(\mathfrak{h})$-valued bilinear form on $U(\mathfrak{g})$ defined by

$$S(g_1, g_2) = \gamma(a\omega(g_1)g_2), \qquad g_1,\ g_2 \in U(\mathfrak{g}).$$

Using the identification of $U(\mathfrak{h})$ with $\mathbb{C}[\mathfrak{h}^*]$, the polynomials on $\mathfrak{h}^*$, any $\mu \in \mathfrak{h}^*$ defines a $\mathbb{C}$-valued symmetric form S_μ on $U(\mathfrak{g})$ defined by evaluation at μ.

For a $\mathfrak{g}$-module V and $\nu \in \mathfrak{h}^*$, let $V[\nu] = \{u \in V : hu = \nu(h)u \text{ for } h \in \mathfrak{h}\}$ denote the weight ν subspace. We consider modules with weight decomposition, so $V = \otimes_\nu V[\nu]$. A singular vector of weight ν in V is $u \in V[\nu]$ such that $\mathfrak{n}_+ u = 0$. The space of all such vectors is denoted $\operatorname{Sing} V[\nu]$. For $\mu \in \mathfrak{h}^*$, denote by M_μ the Verma module with highest weight μ, generated by the vector $\mathbf{1}_\mu \in \operatorname{Sing} V[\mu]$.

We define the Shapovalov form on a Verma module M_μ by

$$S(g_1\mathbf{1}_\mu, g_2\mathbf{1}_\mu) = S_\mu(g_1, g_2), \qquad g_1,\ g_2 \in U(\mathfrak{g}).$$

Weight subspaces $M_\mu[\nu]$ and $M_\mu[\eta]$ are orthogonal for $\nu \neq \eta$ with respect to this form. Let M_μ^* denote the restricted dual module to M_μ, with $\mathfrak{g}$ acting by $(g\phi)(v) = \phi(a(g)v)$ for $g \in \mathfrak{g}$, $\phi \in M_\mu^*$ and $v \in M_\mu$. The Shapovalov form induces a map $S_\mu : M_\mu \to M_\mu^*$ with $S_\mu(\mathbf{1}_\mu) = \mathbf{1}_\mu^*$, where $\mathbf{1}_\mu^*$ is the lowest weight vector dual to $\mathbf{1}_\mu$, and with $S_\mu(g\mathbf{1}_\mu) = \omega(g)\mathbf{1}_\mu^*$ for $g \in \mathfrak{g}$.

The kernel of the Shapovalov form is the maximal proper submodule of M_μ. Denote by L_μ the irreducible quotient. The Shapovalov form induces a bilinear form on L_μ. We shall use the Shapovalov form on a tensor product, defined as the product of the Shapovalov forms of factors.

For $\alpha \in \Delta_+$ and $k \in \{1, 2, \dots\}$, let

$$\chi_k^\alpha(\mu) = (\alpha, \mu + \rho) - \frac{k}{2}(\alpha, \alpha). \tag{2.1}$$

On the weight subspace $U(\mathfrak{n}_-)[-\nu] \subset U(\mathfrak{n}_-)$ the Shapovalov form has determinant [Sh]

$$\det S_\mu[-\nu] = \text{const} \prod_{\alpha \in \Delta_+} \prod_{k = \mathbb{N}} \chi_k^\alpha(\mu)^{P(\nu - k\alpha)},$$

where $S_\mu[-\nu]$ denotes the restriction to $U(\mathfrak{n}_-)[-\nu]$, $\mathbb{N}$ are the numbers $\{1, 2, \dots\}$, $P(\lambda)$ is the Kostant partition function and const is a nonzero constant depending on choice of basis. This formula also holds for $S_\mu[-\nu]$ defined as the restriction to $M_\mu[\mu - \nu]$, so the Shapovalov form on M_μ is nondegenerate if

$$\chi_k^\alpha(\mu) \neq 0 \qquad \text{for all } \alpha \in \Delta_+,\ k \in \{1, 2, \dots, \}. \tag{2.2}$$

Let $S_\mu^{-1}[-\nu]$ be the inverse matrix to $S_\mu[-\nu]$.

Lemma 2.1. *Let $\{F_j\}$ be a basis of $U(\mathfrak{n}_-)[-\nu]$ with the first N elements of $\{F_j \mathbf{1}_\mu\}$ form a basis of* $\mathrm{Ker}(S_\mu[-\nu])$. *Then an entry $(S_\lambda^{-1}[-\nu])_{j\ell}$ of $S_\lambda^{-1}[-\nu]$ is regular at $\lambda = \mu$ if $j > N$ or $\ell > N$.*

Proof. For μ such that $\chi_k^\alpha(\mu)$ is nonzero for every $\alpha \in \Delta_+$, $k \in \mathbb{N}$, the Shapovalov form is invertible, and each $(S_\lambda^{-1}[-\nu])_{j\ell}$ is regular. We next consider μ such that $\chi_k^\alpha(\mu) = 0$ for exactly one $\alpha \in \Delta_+$ and $k \in \mathbb{N}$. There is a unique proper submodule $M_{\mu - k\alpha}$ of highest weight $\mu - k\alpha$, with the dimension of the weight subspace $M_{\mu-k\alpha}[\mu - \nu]$ equal to $P(\nu - k\alpha)$. For $\lambda = \mu + \epsilon\eta$ approaching μ transversely to $\chi_k^\alpha(\lambda) = 0$, the first $N = P(\nu - k\alpha)$ rows of $S_\lambda[-\nu]$ are divisible by ϵ. Any minor $C_{j\ell}$ of $S_\lambda[-\nu]$ with $j > N$ is divisible by ϵ^N, and since the determinant of $S_\lambda[-\nu]$ is divisible by exactly the Nth power of ϵ, the entry $(S_\lambda^{-1}[-\nu])_{j\ell}$ with $j > N$ is regular at $\epsilon = 0$. By the symmetry of S, the entries with $\ell > N$ are also regular.

For general μ, we note that if $\{F_j\}$ is such that the first N elements $\{F_j \mathbf{1}_\mu\}$ form a basis of $\mathrm{Ker}(S_\mu[-\nu])$, then for λ in a neighborhood of μ, $\mathrm{Ker}(S_\lambda[-\nu])$ is contained in the space spanned by the first N elements of $\{F_j \mathbf{1}_\lambda\}$. Then we have that for $j > N$ or $\ell > N$, the meromorphic function $(S_\lambda^{-1}[-\nu])_{j\ell}$ is regular in a neighborhood of μ except perhaps on a subspace of codimension 2. Thus it is regular in the entire neighborhood. □

The proof of this lemma was communicated to the authors by K. Styrkas. It mirrors the proof in [ES] that entries of $S_\mu^{-1}[-\nu]$ can have only simple poles.

2.3. **Singular vectors.** For $\mu \in \mathfrak{h}^*$, let $I_\mu \subset U(\mathfrak{n}_+)$ denote the annihilating ideal of the vector $\mathbf{1}_\mu^* \in M_\mu^*$. Thus $\omega(I_\mu)\mathbf{1}_\mu$ equals $\mathrm{Ker}(S_\mu)$ in M_μ. Let V be a $\mathfrak{g}$ module such that for every $u \in V$, $\mathfrak{n}_+ u$ is finite dimensional. Denote by $V[\nu]_\mu \subset V[\nu]$ the subspace annihilated by I_μ. Let $\{F_j : j \geqslant 0\}$ be a homogeneous basis of $U(\mathfrak{n}_-)$ with $F_0 = \mathbf{1}$, the identity element.

Proposition 2.2. [ES] *There exists a singular vector in $M_\mu \otimes V[\mu + \nu]$ of the form $\mathbf{1}_\mu \otimes u + \sum_{j>0} F_j \mathbf{1}_\mu \otimes u_j$ for some $u_j \in V$ only if u belongs to $V[\nu]_\mu$.*

Proof. The vector $\mathbf{1}_\mu \otimes u + \sum_{j>0} F_j \mathbf{1}_\mu \otimes u_j$ induces a map $M_\mu^* \to V$ defined by

$$\phi \mapsto \phi(\mathbf{1}_\mu)u + \sum_{j>0} \phi(F_j \mathbf{1}_\mu)u_j$$

for $\phi \in M_\mu^*$. Since the vector is singular, this map commutes with the action of $\mathfrak{n}_+$. Clearly, it maps the lowest weight vector $\mathbf{1}_\mu^*$ to u. The existence of such a map implies that $I_\mu u$ is zero. □

For $u \in V[\nu]$, set

$$\Xi(\mu)(\mathbf{1} \otimes u) = \sum_{j,k \geqslant 0} (S_\mu^{-1})_{j\ell} F_j \otimes \omega(F_k)u$$

in $U(\mathfrak{n}_-) \otimes V$. For μ satisfying (2.2), this vector is well-defined since S_μ is non-degenerate.

Proposition 2.3. *For $u \in V[\nu]_\mu$, the vector $\Xi(\lambda)(\mathbf{1} \otimes u) \in U(\mathfrak{n}_-) \otimes V$ as a function of λ is regular at $\lambda = \mu$.*

Proof. We consider a homogenous basis $\{F_j\}$ of $U(\mathfrak{n}_-)$ that contains a subset $\{\tilde{F}_j\}$ such that $\{\tilde{F}_j \mathbf{1}_\mu\}$ forms a basis of $\mathrm{Ker}(S_\mu)$. Since u is in $V[\nu]_\mu$, any $\omega(\tilde{F}_j)u$ equals zero. By Lemma 2.1, the nonzero terms of $\Xi(\lambda)(\mathbf{1} \otimes u)$ are regular at $\lambda = \mu$. □

For $u \in V[\nu]_\mu$, the vector $\Xi(\mu)(\mathbf{1}_\mu \otimes u) \in M_\mu \otimes V[\mu + \nu]$ is singular [ES].

Proposition 2.4. *For $u \in V[\nu]_\mu$, let $\mathbf{1}_\mu \otimes u + \sum_{j>0} F_j \mathbf{1}_\mu \otimes u_j$ be any singular vector. Then the difference*

$$\left(\mathbf{1}_\mu \otimes u + \sum_{j>0} F_j \mathbf{1}_\mu \otimes u_j\right) - \Xi(\mu)(\mathbf{1}_\mu \otimes u)$$

lies in $\mathrm{Ker}(S_\mu) \otimes V$.

Proof. The difference is a singular vector, and the associated map $M_\mu^* \to V$ maps $\mathbf{1}_\mu^* \mapsto 0$. Since this map commutes with $\mathfrak{n}_+$, all of $U(\mathfrak{n}_+)\mathbf{1}_\mu^*$ maps to zero. The image of S_μ in M_μ^* is $U(\mathfrak{n}_+)\mathbf{1}_\mu^*$, so any vector of $M_\mu \otimes V$ orthogonal to it must belong to $\mathrm{Ker}(S_\mu) \otimes V$. □

We define the linear map $\mathcal{Q}(\mu) : V[\nu]_{\mu-\rho-\frac{1}{2}\nu} \to V[\nu]$ as

$$\mathcal{Q}(\mu)u = \sum_{j,k \geqslant 0} \left(S^{-1}_{\mu-\rho-\frac{1}{2}\nu}\right)_{jk} a(F_j)\omega(F_k)u.$$

3. Gaudin and KZB operators

Let $z = (z_1, \dots, z_n)$ be a collection of distinct complex numbers. Let $V_1, \dots, V_n$ be $\mathfrak{g}$-modules, and let $V = V_1 \otimes \cdots \otimes V_n$. For $x \in \mathrm{End}(V_p)$, let $x^{(p)} = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$ be the endomorphism of V with g acting on the pth factor. For $\sum_j x_j \otimes y_j \in \mathrm{End}(V_p) \otimes \mathrm{End}(V_s)$, let $(\sum_j x_j \otimes y_j)^{(p,s)}$ denote $\sum_j x_j^{(p)} y_j^{(s)}$.

Let $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ be the symmetric invariant tensor dual to $(\,,\,)$. It has a decomposition $\Omega = \Omega_0 + \sum_{\alpha \in \Delta} \Omega_\alpha$ where $\Omega_0 \in \mathfrak{h} \otimes \mathfrak{h}$ and $\Omega_\alpha \in \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$. Explicitly, for $\{h_\nu\}$ an orthonormal basis of $\mathfrak{h}$, $\Omega_0 = \sum_\nu h_\nu \otimes h_\nu$, and for generators $e_\alpha \in \mathfrak{g}_\alpha$ with $(e_\alpha, e_{-\alpha}) = 1$, $\Omega_\alpha = e_\alpha \otimes e_{-\alpha}$. The Casimir element is $C = C_0 + \sum_{\alpha \in \Delta} C_\alpha \in U(\mathfrak{g})$, where $C_0 = \sum_\nu h_\nu h_\nu$ and $C_\alpha = e_\alpha e_{-\alpha}$.

3.1. **Rational Gaudin operators.** The rational Gaudin operators are linear operators on V, given by

$$K_p(z) = \sum_{s\neq p} \frac{\Omega^{(p,s)}}{z_p - z_s}, \qquad p = 1, \dots, n.$$

Each $K_p(z)$ commutes with the action of $\mathfrak{g}$ on V. For all p, s, we have $[K_p(z), K_s(z)] = 0$.

3.2. **Trigonometric Gaudin operators.** Let

$$\Omega_+ \;=\; \frac{1}{2}\Omega_0 + \sum_{\alpha\in\Delta_+} \Omega_\alpha \;, \qquad \Omega_- \;=\; \frac{1}{2}\Omega_0 + \sum_{\alpha\in\Delta_-} \Omega_\alpha .$$

The trigonometric r-matrix is defined by

$$r(x) = \frac{\Omega_+ x + \Omega_-}{x-1}.$$

For $\xi \in \mathfrak{h}$, the trigonometric Gaudin operators are defined as

$$\mathcal{K}_p(z,\xi) = \xi^{(p)} + \sum_{s\neq p} r^{(p,s)}(z_p/z_s), \qquad p = 1, \dots, n.$$

Each $\mathcal{K}_p(z,\xi)$ commutes with the action of $\mathfrak{h}$ on V and $[\mathcal{K}_p(z,\xi), \mathcal{K}_s(z,\xi)] = 0$ for all p, s, see [Ch, EFK].

3.3. **KZB operators.** Let $H_+ \subset \mathbb{C}$ be the upper half plane, and $\tau \in H_+$. Let $z_1, \dots, z_n \in \mathbb{C}$ be distinct modulo the lattice $\mathbb{Z} + \tau\mathbb{Z}$. Let $\lambda \in \mathfrak{h}$, with coordinates $\lambda = \sum \lambda_\nu h_\nu$ where (h_ν) is an orthonormal basis of $\mathfrak{h}$. For given z, τ, the KZB operators H_0, H_p are operators acting on functions $u(\lambda)$ with values in $V[0] = (V_1 \otimes \cdots \otimes V_n)[0]$, see [FW].

The KZB operators are

$$H_0(z,\tau) = (4\pi i)^{-1} \triangle + \sum_{p,s} \frac{1}{2} \Gamma_\tau^{(p,s)}(\lambda, z_p - z_s, \tau),$$

$$H_p(z,\tau) = -\sum_\nu h_\nu^{(p)} \partial_{\lambda_\nu} + \sum_{s:s\neq p} \Gamma_z^{(p,s)}(\lambda, z_p - z_s, \tau), \qquad p = 1, \dots, n.$$

Here $\triangle = \sum_v \partial^2_{\lambda_\nu}$ is the Laplace operator and the operators $\Gamma_\tau(\lambda, z, \tau)$, $\Gamma_z(\lambda, z, \tau)$ are defined as follows. For the Jacobi theta function

$$\theta_1(t,\tau) = 2 e^{\frac{\pi i}{4}\tau} \sum_{j=0}^{\infty} (-1)^j e^{\pi i j(j+1)\tau} \sin((2j+1)\pi t)$$

and the functions

$$\rho(t,\tau) = \frac{\theta_1'(t,\tau)}{\theta_1(t,\tau)}, \qquad \sigma_w(t,\tau) = \frac{\theta_1(w-t,\tau)\theta_1'(0,\tau)}{\theta_1(w,\tau)\theta_1(t,\tau)},$$

$$\eta(t,\tau) = \rho(t,\tau)^2 + \rho'(t,\tau), \qquad \varphi(w,t,\tau) = \partial_w \sigma_w(t,\tau).$$

we set

$$\Gamma_\tau(\lambda, z, \tau) = \frac{1}{4\pi i} \eta(z,\tau)\Omega_0 - \frac{1}{2\pi i} \sum_{\alpha\in\Delta} \varphi(\alpha(\lambda), z, \tau)\Omega_\alpha,$$

$$\Gamma_z(\lambda,z,\tau)=\rho(z,\tau)\Omega_0+\sum_{\alpha\in\Delta}\sigma_{\alpha(\lambda)}(z,\tau)\Omega_\alpha,$$

so we have

$$H_0(z,\tau)=\frac{1}{4\pi i}\triangle+\frac{1}{4\pi i}\sum_{p,s}\left[\frac{1}{2}\eta(z_p-z_s,\tau)\Omega_0^{(p,s)}-\sum_{\alpha\in\Delta}\varphi(\alpha(\lambda),z_p-z_s,\tau)\Omega_\alpha^{(p,s)}\right],$$

$$H_p(z,\tau)=-\sum_\nu h_\nu^{(p)}\partial_{\lambda_\nu}+\sum_{s:s\neq p}\left[\rho(z_p-z_s,\tau)\Omega_0^{(p,s)}+\sum_{\alpha\in\Delta}\sigma_{\alpha(\lambda)}(z_p-z_s,\tau)\Omega_\alpha^{(p,s)}\right].$$

By [FW] the operators $H_0(z,\tau),H_1(z,\tau),\dots,H_n(z,\tau)$ commute.

3.4. **Trigonometric KZB operators.** The trigonometric KZB operators are the limits of the operators $H_0(z,\tau),H_1(z,\tau),\dots,H_n(z,\tau)$ as $\tau\to i\infty$.

Proposition 3.1. *The trigonometric KZB operators are*

$$H_0=\frac{1}{4\pi i}\triangle-\frac{1}{4\pi i}\sum_{\alpha\in\Delta_+}\frac{\pi^2}{\sin^2(\pi\alpha(\lambda))}\left(e_\alpha e_{-\alpha}+e_{-\alpha}e_\alpha\right),$$

$$H_p(z)=-\sum_\nu h_\nu^{(p)}\partial_{\lambda_\nu}+\pi\sum_{s:s\neq p}\left[\cot(\pi(z_p-z_s))\Omega^{(p,s)}-\sum_{\alpha\in\Delta_+}\cot(\alpha(\lambda))(\Omega_\alpha^{(p,s)}-\Omega_{-\alpha}^{(p,s)})\right]$$

for $p=1,\dots,n$.

Note that H_0 does not depend on z.

Proof. We have that $\theta_1(t,\tau)=2e^{\frac{\pi}{4}i\tau}\left[\sin(\pi t)+O(e^{2\pi i\tau})\right]$, so as $\tau\to i\infty$

$$\rho(t,\tau)\to\frac{\pi\cos(\pi t)}{\sin(\pi t)},\qquad \sigma_w(t,\tau)\to\frac{\pi\sin(\pi(w-t))}{\sin(\pi w)\sin(\pi t)},$$

$$\eta(t,\tau)\to\frac{\pi^2\cos^2(\pi t)}{\sin^2(\pi t)}-\frac{\pi^2}{\sin^2(\pi t)}=-\pi^2,$$

$$\varphi(w,t,\tau)\to\frac{\pi^2}{\sin(\pi t)}\frac{\sin(\pi(w-w+t)}{\sin^2(\pi w)}=\frac{\pi^2}{\sin^2(\pi w)}.$$

Thus, we have

$$H_0=\frac{1}{4\pi i}\triangle+\frac{1}{4\pi i}\sum_{p,s}\left[\frac{-\pi^2}{2}\Omega_0^{(p,s)}-\sum_{\alpha\in\Delta}\frac{\pi^2}{\sin^2(\pi\alpha(\lambda))}\Omega_\alpha^{(p,s)}\right].$$

Since $\sum_{p,s}\Omega_0^{(p,s)}$ acts as zero on $V[0]$, we have that

$$\begin{aligned}H_0&=\frac{1}{4\pi i}\triangle-\frac{1}{4\pi i}\sum_{p,s}\sum_{\alpha\in\Delta}\frac{\pi^2}{\sin^2(\pi\alpha(\lambda))}\Omega_\alpha^{(p,s)}\\&=\frac{1}{4\pi i}\triangle-\frac{1}{4\pi i}\sum_{\alpha\in\Delta_+}\sum_{p,s}\frac{\pi^2}{\sin^2(\pi\alpha(\lambda))}\left(\Omega_\alpha^{(p,s)}+\Omega_{-\alpha}^{(p,s)}\right).\end{aligned}$$

This gives

$$H_0 = \frac{1}{4\pi i}\triangle - \frac{1}{4\pi i}\sum_{\alpha\in\Delta_+} \frac{\pi^2}{\sin^2(\pi\alpha(\lambda))}\left(e_\alpha e_{-\alpha} + e_{-\alpha}e_\alpha\right)$$

by summing over p and s.

For $p = 1, \dots, n$,

$$H_p(z) = -\sum_\nu h_\nu^{(p)}\partial_{\lambda_\nu} + \sum_{s:s\neq p}\left[\frac{\pi\cos(\pi(z_p - z_s))}{\sin(\pi(z_p - z_s))}\Omega_0^{(p,s)} + \sum_{\alpha\in\Delta}\frac{\pi\sin(\pi(\alpha(\lambda) - z_p + z_s))}{\sin(\pi\alpha(\lambda))\sin(\pi(z_p - z_s))}\Omega_\alpha^{(p,s)}\right].$$

$$= -\sum_\nu h_\nu^{(p)}\partial_{\lambda_\nu} + \pi\sum_{s:s\neq p}\left[\cot(\pi(z_p - z_s))\Omega^{(p,s)} - \sum_{\alpha\in\Delta_+}\cot(\pi\alpha(\lambda))(\Omega_\alpha^{(p,s)} - \Omega_{-\alpha}^{(p,s)})\right]$$

as desired. $\square$

For $s = 1, \dots, n$, let Z_s denote $e^{-2\pi i z_s}$. For $\beta \in \mathfrak{h}^*$, and $\lambda \in \mathfrak{h}$, let $X_\beta(\lambda)$ denote $e^{-2\pi i\beta(\lambda)}$. Then for $p = 1, \dots, n$, we may write

$$H_p(z) = -\sum_\nu h_\nu^{(p)}\partial_{\lambda_\nu} - \pi i\sum_{s\neq p}\left[\frac{Z_p + Z_s}{Z_p - Z_s}\Omega^{(p,s)} + \sum_{\alpha\in\Delta_+}\frac{1 + X_\alpha}{1 - X_\alpha}\left(\Omega_\alpha^{(p,s)} - \Omega_{-\alpha}^{(p,s)}\right)\right]. \tag{3.1}$$

3.5. **KZ and KZB equations.** The rational, trigonometric Gaudin operators and the KZB operators are the right hand sides of the rational, trigonometric KZ equations and the KZB equations respectively. Let κ be a nonzero complex number. Then the rational KZ equations for a V-valued function $u(z)$, $z \in \mathbb{C}^n$, are

$$\kappa\partial_{z_p}u(z) = K_p(z)u(z), \qquad p = 1, \dots, n.$$

The trigonometric KZ equations for a V-valued function $u(z)$ are

$$\kappa z_p\partial_{z_p}u(z) = \mathcal{K}_p(z, \xi)u(z), \qquad p = 1, \dots, n.$$

The KZB equations for a $V[0]$-valued function $u(\lambda, z, \tau)$ are

$$\kappa\partial_{z_p}u(\lambda, z, \tau) = H_p(z, \tau)u(\lambda, z, \tau), \qquad p = 1, \dots, n,$$

$$\kappa\partial_\tau u(\lambda, z, \tau) = H_0(z, \tau)u(\lambda, z, \tau),$$

see [KZ, Ch, EFK, FW].

4. Relations among Gaudin and KZB operators

4.1. **Rational Gaudin and trigonometric Gaudin operators.** Let $V_1, \dots, V_n$ be highest weight $\mathfrak{g}$-modules, with highest weights $\Lambda_1, \dots, \Lambda_n$ respectively, and $V = V_1 \otimes \dots \otimes V_n$. Let M_μ be the Verma module with highest weight μ generated by $\mathbf{1}_\mu$.

We label the factors in $M_\mu \otimes V$ starting with zero, so that the pth factor is V_p for $p = 1, \dots, n$. Then as a consequence of Proposition 2.1 in [MaV], we have the following fact.

Proposition 4.1. *For $\mu \in \mathfrak{h}^*$, $u \in V[\nu]_\mu$, and $p = 1, \ldots, n$,*

$$\left(z_p K_p(0, z_1, \ldots, z_n) + \frac{1}{2}(\Lambda_p, \Lambda_p + 2\rho)\right)\Xi(\mu)(\mathbf{1}_\mu \otimes u)$$
$$= \Xi(\mu)\left(\mathbf{1}_\mu \otimes \mathcal{K}_p\left(z_1, \ldots, z_n, \mu + \rho + \frac{\nu}{2}\right) u\right)$$

holds in $\operatorname{Sing} M_\mu \otimes V[\nu + \mu]$.

Proof. [MaV] The left side belongs to $\operatorname{Sing} M_\mu \otimes V[\nu + \mu]$ since $K_p(0, z_1, \ldots, z_n)$ commutes with $\mathfrak{g}$. We rewrite the left side as

$$\left(\Omega^{(p,0)} + \sum_{s \neq 0,p} \frac{z_p \Omega^{(p,s)}}{z_p - z_s} + \frac{1}{2}C^{(p)}\right)\Xi(\mu)(\mathbf{1}_\mu \otimes u)$$

using the fact that the Casimir element applied to V_p is multiplication by the constant $(\Lambda_p, \Lambda_p + 2\rho)$. We have that $\frac{z_p \Omega^{(p,s)}}{z_p - z_s} = r^{(p,s)}(z_p/z_s) + \Omega_-^{(p,s)}$. We define $C_\pm = \frac{1}{2}C_0 + \sum_{\alpha \in \Delta_\pm} C_\alpha$, so $C = C_+ + C_-$ and set $\Omega_\pm^{(p,p)} = C_\pm^{(p)}$, which is just a choice of ordering. Then the previous expression is equal to

$$\left(\sum_{s \neq 0,p} r^{(p,s)}(z_p/z_s) + \sum_{s=0}^{n} \Omega_-^{(p,s)} + \Omega_+^{(p,0)} + \frac{1}{2}\left(C_+^{(p)} - C_-^{(p)}\right)\right)\Xi(\mu)(\mathbf{1}_\mu \otimes u).$$

We note that $\sum_{s=0}^n \Omega_-^{(p,s)} = \frac{1}{2}\sum_{s=0}^n \Omega_0^{(p,s)} + \sum_{\alpha \in \Delta_+} e_{-\alpha}^{(p)}\left(\sum_{s=0}^n e_\alpha^{(s)}\right)$. Since $\Xi(\mu)(\mathbf{1}_\mu \otimes u)$ is singular, $\left(\sum_{s=0}^n e_\alpha^{(s)}\right)\Xi(\mu)(\mathbf{1}_\mu \otimes u) = 0$. Furthermore, $\frac{1}{2}\left(\sum_{s=0}^n \Omega_0^{(p,s)}\right)\Xi(\mu)(\mathbf{1}_\mu \otimes u)$ is just $\frac{1}{2}(\nu + \mu)^{(p)}\Xi(\mu)(\mathbf{1}_\mu \otimes u)$. We also note that $C_+ - C_- = \sum_{\alpha \in \Delta_+}[e_\alpha, e_{-\alpha}]$ equals 2ρ. We are left with

$$\left(\sum_{s \neq 0,p} r^{(p,s)}(z_p/z_s) + \frac{1}{2}(\nu + \mu)^{(p)} + \rho^{(p)} + \Omega_+^{(p,0)}\right)\Xi(\mu)(\mathbf{1}_\mu \otimes u). \tag{4.1}$$

For μ satisfying (2.2), each singular vector has an expansion $\mathbf{1}_\mu \otimes v + \sum_{j,k>0}(S_\mu^{-1})_{jk} F_j \otimes \omega(F_k)(\mathbf{1}_\mu \otimes v)$ for some nonzero $v \in V[\nu]$. We must determine v and can ignore the higher order terms. The first three terms of (4.1) do not act on the first factor M_μ at all. For the last, recall that $\Omega_+^{(p,0)} = \frac{1}{2}\Omega_0^{(p,0)} + \sum_{\alpha \in \Delta_+} e_\alpha^{(p)} e_{-\alpha}^{(0)}$, hence, $\Omega_+^{(p,0)}\Xi(\mu)(\mathbf{1}_\mu \otimes u) = \frac{1}{2}\mu^{(p)}\mathbf{1}_\mu \otimes u +$ *higher order terms*. So (4.1) equals

$$\Xi(\mu)(\mathbf{1}_\mu \otimes \left(\sum_{s \neq 0,p} r^{(p,s)}(z_p/z_s) + \frac{1}{2}\nu^{(p)} + \mu^{(p)} + \rho^{(p)}\right) u)$$

as desired.

For any μ and $u \in V[\nu]_\mu$, we note that

$$\left(\sum_{s \neq 0,p} r^{(p,s)}(z_p/z_s) + \frac{1}{2}(\nu + \lambda)^{(p)} + \rho^{(p)} + \Omega_+^{(p,0)}\right)\Xi(\lambda)(\mathbf{1} \otimes u)$$

is a $U(\mathfrak{h} \oplus \mathfrak{n}_-) \otimes V$-valued rational function of $\mathfrak{h}^*$ regular at $\lambda = \mu$. It is equal, modulo the annihilator of $\mathbf{1}_\lambda$, to

$$\Xi(\lambda)\left(\mathbf{1} \otimes \mathcal{K}_p\left(z_1, \ldots, z_n, \lambda + \rho + \frac{\nu}{2}\right) u\right),$$

so

$$\Xi(\mu)\left(\mathbf{1}_\mu \otimes \mathcal{K}_p\left(z_1, \ldots, z_n, \mu + \rho + \frac{\nu}{2}\right) u\right)$$

is defined, and the proposition holds. □

Corollary 4.2. *For $\mu \in \mathfrak{h}^*$, and $p = 1, \ldots, n$, the operator $\mathcal{K}_p\left(z, \mu + \rho + \frac{\nu}{2}\right)$ preserves $V[\nu]_\mu$.*

Proof. Since $\mathcal{K}_p(z, \mu + \rho + \frac{\nu}{2})$ commutes with $\mathfrak{h}$, we need only check that

$$\Xi(\mu)\left(\mathbf{1}_\mu \otimes \mathcal{K}_p\left(z, \mu + \rho + \frac{\nu}{2}\right) u\right)$$

is well-defined for $u \in V[\nu]_\mu$. By Proposition 4.1, it equals

$$\left(z_p K_p(0, z_1, \ldots, z_n) + \frac{1}{2}(\Lambda_p, \Lambda_p + 2\rho)\right) \Xi(\mu)(\mathbf{1}_\mu \otimes u)$$

which is well-defined. □

Corollary 4.3. *Let $\mathbf{1}_\mu \otimes u + \sum_{j>0} F_j \mathbf{1}_\mu \otimes u_j \in \operatorname{Sing} M_\mu \otimes V[\mu + \nu]$ be an eigenvector of $K_p(0, z_1, \ldots, z_p)$ with eigenvalue ε_p for $p = 1, \ldots, n$. Then the leading term $u \in V[\nu]$ is an eigenvector of $\mathcal{K}_p(z_1, \ldots, z_n, \mu + \rho + \frac{\nu}{2})$ with eigenvalue $z_p \varepsilon_p + \frac{1}{2}(\Lambda_p, \Lambda_p + 2\rho)$.*

Proof. By Proposition 2.2, u belongs to $V[\nu]_\mu$. By Proposition 2.3, the vector $\Xi(\mu)(\mathbf{1}_\mu \otimes u)$ is well-defined. By Proposition prop:singdifference, the difference

$$\tilde{u} = \left(\mathbf{1}_\mu \otimes u + \sum_{j>0} F_j \mathbf{1}_\mu \otimes u_j\right) - \Xi(\mu)(\mathbf{1}_\mu \otimes u)$$

belongs to $\operatorname{Sing} \operatorname{Ker}(S_\mu) \otimes V[\mu + \nu]$. Since $U(\mathfrak{g})$ preserves $\operatorname{Ker}(S_\mu)$, we have

$$\left(z_p K_p(0, z_1, \ldots, z_n) + \frac{1}{2}(\Lambda_p, \Lambda_p + 2\rho)\right) \tilde{u} \in \operatorname{Ker}(S_\mu) \otimes V$$

so the leading term is zero. By Proposition 4.1, this leading term is

$$\left(z_p \varepsilon_p + \frac{1}{2}(\Lambda_p, \Lambda_p + 2\rho)\right) - \mathcal{K}_p\left(z_1, \ldots, z_n, \mu + \rho + \frac{\nu}{2}\right).$$

□

4.2. **Eigenfunctions of KZB operators in the trigonometric limit.** We look for eigenfunctions to H_0 in the space $\mathcal{A}(\xi) \otimes V[0]$ defined as follows.

For $\beta \in \mathfrak{h}^*$, let $X_\beta(\lambda) = e^{-2\pi i \beta(\lambda)}$. For a simple root α_j, let $X_j = X_{\alpha_j}$. Denote by $\mathcal{A}$ the algebra of functions on $\mathfrak{h}$ which can be expressed as meromorphic functions of variables $X_1, \ldots, X_r$ with poles only on $\cup_{\alpha \in \Delta}\{X_\alpha = 1\}$. For $\xi \in \mathfrak{h}^*$, we define $\mathcal{A}(\xi)$ to be the vector space of functions of the form $e^{2\pi i \xi(\lambda)} \phi$ with $\phi \in \mathcal{A}$.

The algebra $\mathcal{A}$ is a subalgebra of the algebra $\mathbb{C}[[X_1, \ldots, X_r]]$ of formal power series. For $\phi(X_1, \ldots, X_r) \in \mathcal{A}$, we define its leading term to be $\mathrm{const}_{\{X_j\}}(\phi) = \phi(0, \ldots, 0)$. For $e^{2\pi i \xi(\lambda)}\phi \in \mathcal{A}(\xi)$, we define its leading term to be $\phi(0, \ldots, 0)$.

Let $E(\xi) \subset \mathcal{A}(\xi) \otimes V[0]$ be the subspace of eigenvectors to H_0 with eigenvalue $\pi i(\xi, \xi)$.

Proposition 4.4. [FV2] *Let $\xi \in \mathfrak{h}$ satisfy*

$$(\xi - \beta, \xi - \beta) \neq (\xi, \xi) \text{ for all nonzero } \beta \in Q_+ \ . \tag{4.2}$$

Then for $u \in V[0]$, there exists a unique $\psi_u^\xi = e^{2\pi i \xi(\lambda)}\phi_u^\xi \in \mathcal{A}(\xi) \otimes V[0]$ such that

$$H_0 \psi_u^\xi = \pi i(\xi, \xi)\psi_u^\xi \quad \text{and} \quad \mathrm{const}_{\{X_j\}}(\phi_u^\xi) = u.$$

The proof of existence and uniqueness of ψ_u^ξ as a formal power series is similar to Heckman and Opdam [HO]. Etingof [E] gave a representation theoretic construction of eigenfunctions to H_0 for ξ satisfying (4.2). Let $M_{\xi-\rho}$ be the Verma module of highest weight $\xi - \rho$, where ξ satisfies (2.2). Then for each $u \in V[0]$, there is a unique homomorphism $\Phi_u : M_{\xi-\rho} \to M_{\xi-\rho} \otimes V$ such that $\Phi_u(\mathbf{1}_{\xi-\rho}) = \Xi(\xi - \rho)(\mathbf{1}_{\xi-\rho} \otimes u)$. Then as formal power series,

$$\psi_u^\xi(\lambda) = \frac{\mathrm{tr}_{M_{\xi-\rho}} \Phi_u \exp(2\pi i \lambda)}{\mathrm{tr}_{M_{\xi-\rho}} \exp(2\pi i \lambda)}.$$

Felder and Varchenko [FV2] gave the explicit calculation of this function and showed that it lies in $A(\xi) \otimes V[0]$. We recall this construction.

We first define a linear map $A_X : \ U(\mathfrak{n}_-) \to \mathcal{A} \otimes U(\mathfrak{n}_-)$. We set $A_X(\mathbf{1}) = \mathbf{1}$ for $\mathbf{1}$ the identity element of $U(\mathfrak{n}_-)$.

For an element of $U(\mathfrak{n}_-)$ of the form $F_{\beta_1} \cdots F_{\beta_m}$ where each F_{β_k} belongs to $\mathfrak{g}_{-\beta_k}$ with $\beta_k \in \Delta_+$, we set $A_X(F_{\beta_1} \cdots F_{\beta_m}) = \sum_{\sigma \in S_m} A_X^\sigma(F_{\beta_1} \cdots F_{\beta_m})$, where S_m is the symmetric group on m symbols, and A_X^σ is defined by

$$A_X^\sigma(F_{\beta_1} \cdots F_{\beta_m}) = \prod_{k=1}^m \frac{X_{\beta_{\sigma(k)}}^{a_k^\sigma + 1}}{1 - X_{\beta_{\sigma(1)}} \cdots X_{\beta_{\sigma(k)}}} F_{\beta_{\sigma(1)}} \cdots F_{\beta_{\sigma(m)}}.$$

For given $\sigma \in S_m$, the number a_k^σ is defined to be $\sum_{j=k}^{m-1} d_j^\sigma$, where $d_j^\sigma = 1$ if $\sigma(j) > \sigma(j+1)$ and $d_j^\sigma = 0$ otherwise.

Lemma 4.5. *The operator A_X is well-defined. In other words the relation*

$$A_X(F_{\beta_1} \cdots F_{\beta_\ell} F_{\beta_{\ell+1}} \cdots F_{\beta_m}) - A_X(F_{\beta_1} \cdots F_{\beta_{\ell+1}} F_{\beta_\ell} \cdots F_{\beta_m}) = A_X(F_{\beta_1} \cdots [F_{\beta_\ell} F_{\beta_{\ell+1}}] \cdots F_{\beta_m})$$

holds for any collection $F_{\beta_1}, \ldots, F_{\beta_m}$ with each $\beta_k \in \Delta_+$ and $F_{\beta_k} \in \mathfrak{g}_{-\beta_k}$.

Proof. For each $\sigma \in S_m$, there exists some σ' such that

$$A_X^\sigma(F_{\beta_1} \cdots F_{\beta_\ell} F_{\beta_{\ell+1}} \cdots F_{\beta_m}) - A_X^{\sigma'}(F_{\beta_1} \cdots F_{\beta_{\ell+1}} F_{\beta_\ell} \cdots F_{\beta_m}) = B^\sigma(X) F_{\beta_{\sigma(1)}} \cdots F_{\beta_{\sigma(m)}} \tag{4.3}$$

with $B^\sigma(X) \in \mathcal{A}$. In fact, $\sigma' = \tau_{\ell,\ell+1} \circ \sigma$ where $\tau_{\ell,\ell+1}$ is the transposition of ℓ and $\ell + 1$. It is clear that

$$A_X(F_{\beta_1} \cdots F_{\beta_\ell} F_{\beta_{\ell+1}} \cdots F_{\beta_m}) - A_X(F_{\beta_1} \cdots F_{\beta_{\ell+1}} F_{\beta_\ell} \cdots F_{\beta_m}) = \sum_{\sigma \in S_m} B^\sigma(X) F_{\beta_{\sigma(1)}} \cdots F_{\beta_{\sigma(m)}}.$$

To calculate $B^\sigma(X)$, we note that the denominators of $A_X^\sigma(F_{\beta_1} \cdots F_{\beta_\ell} F_{\beta_{\ell+1}} \cdots F_{\beta_m})$ and $A_X^{\sigma'}(F_{\beta_1} \cdots F_{\beta_{\ell+1}} F_{\beta_\ell} \cdots F_{\beta_m})$ are the same, since these only depend on the order of the F_β

appearing in $F_{\beta_{\sigma(1)}} \cdots F_{\beta_{\sigma(m)}}$ and $F_{\beta_{\sigma'(1)}} \cdots F_{\beta_{\sigma'(m)}}$. The difference between A^σ and $A^{\sigma'}$ comes from the coefficients a_k^σ and $a_k^{\sigma'}$, depending on d_j^σ and $d_j^{\sigma'}$ for $j \leqslant k$.

We first consider σ for which the factors F_{β_ℓ} and $F_{\beta_{\ell+1}}$ are not adjacent in the expression $F_{\beta_{\sigma(1)}} \cdots F_{\beta_{\sigma(m)}}$, in other words, σ for which $|\sigma^{-1}(\ell) - \sigma^{-1}(\ell+1)| > 1$. For such σ, exchanging F_{β_ℓ} and $F_{\beta_{\ell+1}}$ produces no change in the relative sizes of the indices of adjacent factors in $F_{\beta_{\sigma(1)}} \cdots F_{\beta_{\sigma(m)}}$, so $d_j^\sigma = d_j^{\sigma'}$ for all j and $a_k^\sigma = a_k^{\sigma'}$ for all k. This tells us that for these σ, $B^\sigma(X) = 0$.

For the remaining σ, F_{β_ℓ} and $F_{\beta_{\ell+1}}$ are adjacent in $F_{\beta_{\sigma(1)}} \cdots F_{\beta_{\sigma(m)}}$. We first consider σ such that $F_{\beta_\ell} F_{\beta_{\ell+1}}$ appears, or $\sigma^{-1}(\ell) + 1 = \sigma^{-1}(\ell+1)$. We calculate $d^\sigma_{\sigma^{-1}(\ell)} = 0$, since $\sigma(\sigma^{-1}(\ell))$ is less than $\sigma(\sigma^{-1}(\ell)+1)$. We see that $d^{\sigma'}_{\sigma^{-1}(\ell)} = d^{\sigma'}_{\sigma'^{-1}(\ell+1)} = 1$. For all $j \neq \sigma^{-1}(\ell)$, it is clear that $d_j^\sigma = d_j^{\sigma'}$. Thus $a_k^{\sigma'} = a_k^\sigma + 1$ for $k \leqslant \sigma^{-1}(\ell)$ and $a_k^{\sigma'} = a_k^\sigma$ for $k > \sigma^{-1}(\ell)$. For such σ,

$$A_X^{\sigma'}(F_{\beta_1} \cdots F_{\beta_{\ell+1}} F_{\beta_\ell} \cdots F_{\beta_m}) = X_{\beta_{\sigma(1)}} \cdots X_{\beta_{\sigma(\sigma^{-1}(\ell))}} A_X^\sigma(F_{\beta_1} \cdots F_{\beta_\ell} F_{\beta_{\ell+1}} \cdots F_{\beta_m})$$

so

$$B^\sigma(X) F_{\beta_{\sigma(1)}} \cdots F_{\beta_{\sigma(m)}} = (1 - X_{\beta_{\sigma(1)}} \cdots X_{\beta_{\sigma(\sigma^{-1}(\ell))}}) A_X^\sigma(F_{\beta_1} \cdots F_{\beta_\ell} F_{\beta_{\ell+1}} \cdots F_{\beta_m}).$$

We must also consider σ such that $F_{\beta_{\ell+1}} F_{\beta_\ell}$ appears in $F_{\beta_{\sigma(1)}} \cdots F_{\beta_{\sigma(m)}}$. For these σ, we see that σ' is in the previous category. Then $B^\sigma(X) = -B^{\sigma'}(X)$, so

$$B^\sigma(X) F_{\beta_{\sigma(1)}} \cdots F_{\beta_{\sigma(m)}} + B^{\sigma'}(X) F_{\beta_{\sigma'(1)}} \cdots F_{\beta_{\sigma'(m)}} B^\sigma(X) F_{\beta_{\sigma(1)}} \cdots [F_{\beta_\ell}, F_{\beta_{\ell+1}}] \cdots F_{\beta_{\sigma(m)}}.$$

We can write $B^\sigma(X)$ more explicitly with variables Y_{β_k}, where $Y_{\beta_k} = X_{\beta_k}$ for $k < \ell$, $Y_{\beta_\ell} = X_{\beta_\ell} X_{\beta_{\ell+1}}$, and $Y_{\beta_k} = X_{\beta_{k+1}}$ for $k > \ell$. Then

$$B^\sigma(X) = \prod_{k=1}^{m-1} \frac{Y_{\beta_{\widehat{\sigma}(k)}}^{b_k^{\widehat{\sigma}}}}{1 - Y_{\beta_{\widehat{\sigma}(1)}} \cdots Y_{\beta_{\widehat{\sigma}(k)}}}$$

where $\widehat{\sigma}$ is the element of S_{m-1} that acts like σ, but treats ℓ and $\ell+1$ as a unit, and $b_k^{\widehat{\sigma}}$ is a sum of the first k numbers d_j^σ, skipping $d^\sigma_{\sigma^{-1}(\ell)}$, which is zero. In fact,

$$B^\sigma(X) F_{\beta_{\sigma(1)}} \cdots [F_{\beta_\ell}, F_{\beta_{\ell+1}}] \cdots F_{\beta_{\sigma(m)}} = A_Y^{\widehat{\sigma}}(F_{\beta_1} \cdots [F_{\beta_\ell}, F_{\beta_{\ell+1}}] \cdots F_{\beta_m}),$$

and

$$\sum_{\widehat{\sigma} \in S_m} A_Y^{\widehat{\sigma}}(F_{\beta_1} \cdots [F_{\beta_\ell}, F_{\beta_{\ell+1}}] \cdots F_{\beta_m}) = A_Y(F_{\beta_1} \cdots [F_{\beta_\ell}, F_{\beta_{\ell+1}}] \cdots F_{\beta_m}).$$

□

Proposition 4.6. [FV2] *For* $\mathbf{1}_{\xi-\rho} \otimes u + \sum_{j>0} F_j \otimes u_j \in \operatorname{Sing} M_{\xi-\rho} \otimes V[0]$, *the function*

$$\psi(\lambda) = e^{2\pi i \xi(\lambda)}(u + \sum_{j>0} A_X(F_j) u_j)$$

belongs to $E(\xi)$.

For $u \in V[0]_{\xi-\rho}$, we denote the function associated to $\Xi(\xi-\rho)(\mathbf{1}_{\xi-\rho} \otimes u)$ by

$$\psi_u^\xi = e^{2\pi i \xi(\lambda)}(u + \sum_{j>0} (S_{\xi-\rho}^{-1})_{jk} A_X(F_j) \omega(F_k) u_j).$$

Let $P^\xi : V[0]_{\xi-\rho} \to E(\xi)$ denote the map $u \mapsto \psi_u^\xi$.

Proposition 4.7. [FV2] *For $\xi \in \mathfrak{h}^*$ satisfying (4.2) the map P^ξ is an isomorphism between $V[0]$ and $E(\xi)$.*

Proposition 4.8. *For $p = 1, \dots, n$, the operator $H_p(z)$ preserves $E(\xi)$.*

Proof. We use the formulation of $H_p(z)$ in formula (3.1). Differentiations with respect to λ preserve $\mathcal{A}(\xi)$, so $H_p(z)$ preserves $\mathcal{A}(\xi) \otimes V[0]$. Since H_0 and $H_p(z)$ commute, $H_p(z)$ also preserves $E(\xi)$. □

4.3. **Trigonometric Gaudin and trigonometric KZB operators.**

Lemma 4.9. *For ξ with property (4.2) and $p = 1, \dots, n$, we have*

$$H_p(z_1, \dots, z_n) P^\xi = P^\xi \mathcal{K}_p(e^{-2\pi i z_1}, \dots, e^{-2\pi i z_n}, \xi)$$

as maps from $V[0]$ to $E(\xi)$.

Proof. For any $u \in V[0]$, we have $H_p(z)P^\xi(u) = H_p(z)\psi_u^\xi \in E(\xi)$ by Proposition 4.8. Since ξ satisfies (4.2), $H_p(z)\psi_u^\xi$ equals $\psi_v^\xi(\lambda)$ for some $v \in V[0]$. By formula (3.1) we have

$$\begin{aligned}
v &= -2\pi i \xi^{(p)} u - \pi i \sum_{s \neq p} \left[\frac{Z_p + Z_s}{Z_p - Z_s} \Omega^{(p,s)} + \sum_{\alpha \in \Delta_+} \left(\Omega_\alpha^{(p,s)} - \Omega_{-\alpha}^{(p,s)} \right) \right] u \\
&= -2\pi i \xi^{(p)} u - \pi i \sum_{s \neq p} \left[\frac{Z_p + Z_s}{Z_p - Z_s} (\Omega_+^{(p,s)} + \Omega_-^{(p,s)}) + (\Omega_+^{(p,s)} - \Omega_-^{(p,s)}) \right] u \\
&= -2\pi i \xi^{(p)} u - 2\pi i \sum_{s \neq p} r^{(p,s)} \left(\frac{Z_p}{Z_s} \right) u \\
&= -2\pi i \mathcal{K}_p(Z_1, \dots, Z_n, \xi) u.
\end{aligned}$$

□

The lemma holds for generic ξ and so implies the following result.

Corollary 4.10. *For $\xi \in \mathfrak{h}^*$ and $u \in V[0]_{\xi-\rho}$, we have*

$$H_p(z_1, \dots, z_n)\psi_u^\xi = \psi_v^\xi$$

for $v = \mathcal{K}_p(e^{-2\pi i z_1}, \dots, e^{-2\pi i z_n}, \xi)u$.

5. Scalar products

5.1. **Shapovalov form and rational Gaudin operators.** We recall the following fact, see for example [RV].

Proposition 5.1. *Let $V = V_1 \otimes \cdots \otimes V_n$, and let S be the tensor Shapovalov form on V. Let $u, v \in V$. Then*

$$S(K_p(z_1, \dots, z_n)u, v) = S(u, K_p(z_1, \dots, z_n)v), \qquad p = 1, \dots n.$$

5.2. **Scalar product and trigonometric Gaudin operators.** We define the following family of bilinear forms on V, depending on a parameter $\xi \in \mathfrak{h}^*$.

Definition 5.2. *For $\xi, \nu \in \mathfrak{h}^*$ introduce a bilinear form $\langle , \rangle_\xi$ on $V[\nu]_{\xi-\rho-\frac{1}{2}\nu}$ by the formula*

$$\langle u, v\rangle_\xi = S(u, \mathcal{Q}(\xi)v). \tag{5.1}$$

Proposition 5.3. *The bilinear form $\langle\, ,\, \rangle_\xi$ on $V[\nu]_{\xi-\rho-\frac{1}{2}\nu}$ and the Shapovalov form on $M_{\xi-\rho-\frac{1}{2}\nu} \otimes V$ satisfy the relation*

$$\langle u, v\rangle_\xi = S\left(\Xi(\xi-\rho-\frac{1}{2}\nu)(\mathbf{1}_{\xi-\rho-\frac{1}{2}\nu}\otimes u), \Xi(\xi-\rho-\frac{1}{2}\nu)(\mathbf{1}_{\xi-\rho-\frac{1}{2}\nu}\otimes v)\right).$$

Proof. Set $\mu = \xi-\rho-\frac{1}{2}\nu$. Then

$$\Xi(\mu)(\mathbf{1}_\mu \otimes u) = \mathbf{1}_\mu \otimes u + \sum_{j,k>0} (S_\mu^{-1})_{jk}\left(F_j \otimes \omega(F_k)\right)(\mathbf{1}_\mu \otimes u)$$

with respect to a homogeneous basis F_j of $U(\mathfrak{n}_-)$. Thus the right side of the equation (5.1) is

$$\sum_{j,k,l,m\geqslant 0} (S_\mu^{-1})_{jk}(S_\mu^{-1})_{lm} S\left(F_j\mathbf{1}_\mu \otimes \omega(F_k)u, F_l\mathbf{1}_\mu \otimes \omega(F_m)v\right),$$

or

$$\sum_{j,k,l,m\geqslant 0} (S_\mu^{-1})_{jk}(S_\mu^{-1})_{lm}(S_\mu)_{jl} S(\omega(F_k)u, \omega(F_m)v) = \sum_{k,m\geqslant 0} (S_\mu^{-1})_{km} S(u, a(F_k)\omega(F_m)v).$$

This last expression is just $S(u, \mathcal{Q}(\xi)w)$. $\square$

Corollary 5.4. *Let $\mathbf{1}_\mu \otimes u + \sum_{j>0} F_j \otimes u_j$ and $\mathbf{1}_\mu \otimes v + \sum_{j>0} F_j \otimes v_j$ be two vectors in* $\operatorname{Sing} M_\mu \otimes V[\mu+\nu]$. *Then the relation*

$$S\left(\mathbf{1}_\mu \otimes u + \sum_{j>0} F_j \otimes u_j, \mathbf{1}_\mu \otimes v + \sum_{j>0} F_j \otimes v_j\right) = \langle u, v\rangle_{\mu+\rho+\frac{1}{2}\nu}$$

holds.

Proof. The singular vectors differ from $\Xi(\mu)(\mathbf{1}_\mu \otimes u)$ and $\Xi(\mu)(\mathbf{1}_\mu \otimes v)$ by vectors in $\operatorname{Ker}(S_\mu) \otimes V$. $\square$

Corollary 5.5. *The bilinear form $\langle\, ,\, \rangle_\xi$ on $V[\nu]_{\xi-\rho-\frac{1}{2}\nu}$ is symmetric.*

Theorem 5.6. *For $\xi \in \mathfrak{h}^*$, $u, v \in V[\nu]_{\xi-\rho-\frac{1}{2}\nu}$ and $p = 1, \dots, n$, we have*

$$\langle \mathcal{K}_p(z,\xi)u, v\rangle_\xi = \langle u, \mathcal{K}_p(z,\xi)v\rangle_\xi.$$

Proof. Note that both sides are defined by Corollary 4.2. By Proposition 5.1, we know that $z_p K_p(0, z_1, \dots, z_n) + \frac{1}{2}(\Lambda_p, \Lambda_p + 2\rho)$ is symmetric with respect to the Shapovalov form defined on $M_{\xi-\rho-\frac{1}{2}\nu} \otimes V$. This and Proposition 4.1 imply that

$$S(\Xi(\mu)(\mathbf{1}_\mu \otimes \mathcal{K}_p(z,\xi)u), \Xi(\mu)(\mathbf{1}_\mu \otimes v)) = S(\Xi(\mu)(\mathbf{1}_\mu \otimes u), \Xi(\mu)(\mathbf{1}_\mu \otimes \mathcal{K}_p(z,\xi)v))$$

for $\mu = \xi - \rho - \frac{1}{2}\nu$. Then by Proposition 5.3 we have the theorem. $\square$

5.3. **Scalar product and KZB operators.**

Definition 5.7. *For $\xi \in \mathfrak{h}^*$, we define the bilinear form $\langle\, ,\, \rangle$ on $\oplus_{\beta \in Q} \mathcal{A}(\xi + \beta) \otimes V[0]$ by*

$$\langle \psi_1(\lambda), \psi_2(\lambda) \rangle = \int_C S(\psi_1(\lambda), \psi_2(-\lambda)) d\lambda_1 d\lambda_2 \cdots d\lambda_r \tag{5.2}$$

where C is given by each $\lambda_j = (\lambda, \alpha_j)$ ranging along the interval from $0 - i\delta$ to $1 - i\delta$ for some $\delta > 0$.

Proposition 5.8. *The bilinear form $\langle\, ,\, \rangle$ is well-defined on $\oplus_{\beta \in Q} \mathcal{A}(\xi + \beta) \otimes V[0]$.*

Proof. Recall that $\mathcal{A}(\mu)$ is defined as the space of functions on $\mathfrak{h}$ of the form $\psi(\lambda) = e^{2\pi i \mu(\lambda)} \phi(X)$ where $X = (X_1, \ldots, X_r)$, $X_j = e^{-2\pi i \alpha_j(\lambda)}$, and ϕ is a meromorphic function with poles only on the hyperplanes $\cup_{\alpha \in \Delta} \{X_\alpha = 1\}$. For $\psi_1(\lambda) \in \mathcal{A}(\xi) \otimes V[0]$ and $\psi_2(\lambda) \in \mathcal{A}(\xi + \beta) \otimes V[0]$ with $\beta = \sum_{j=1}^r b_j \alpha_j \in Q$ the bilinear form is

$$\langle \psi_1(\lambda), \psi_2(\lambda) \rangle = \int_C S(e^{2\pi i \xi(\lambda)} \phi_1(X), e^{-2\pi i \xi(\lambda)} e^{-2\pi i \beta(\lambda)} \phi_2(X^{-1})) d\lambda_1 d\lambda_2 \cdots d\lambda_r$$

where X^{-1} denotes $(X_1^{-1}, \ldots, X_r^{-1})$. The factor $e^{-2\pi i \beta(\lambda)}$ is $X_1^{b_1} \cdots X_r^{b_r}$, so the integrand is periodic and we may write

$$\langle \psi_1(\lambda), \psi_2(\lambda) \rangle = (-2\pi i)^{-r} \int_{\tilde{C}} X_1^{b_1} \cdots X_r^{b_r} S(\phi_1(X), \phi_2(X^{-1})) \frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_r}{X_r} \tag{5.3}$$

where $\tilde{C}$ is a torus $\{X \mid |X_j| = \epsilon, j = 1, \ldots, r\}$ with $\epsilon < 1$. The torus $\tilde{C}$ doesn't cross the poles of $\phi_1(X)$ or $\phi_2(X^{-1})$. □

Proposition 5.9. *The trigonometric KZB operators $H_0, H_1, \ldots, H_n$ are symmetric with respect to $\langle\, ,\rangle$ on $\oplus_{\beta \in Q} \mathcal{A}(\xi + \beta) \otimes V[0]$.*

Proof. Recall that H_0 is

$$H_0 = \frac{1}{4\pi i} \triangle - \frac{1}{4\pi i} \sum_{\alpha \in \Delta_+} \frac{\pi^2}{\sin^2(\pi \alpha(\lambda))} (e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha).$$

Since the integrand of (5.2) is periodic, the Laplace operator $\triangle$ is symmetric with respect to $\langle\, ,\rangle$. For each α, the operator $e_\alpha e_{-\alpha}$ is adjoint to $e_{-\alpha} e_\alpha$ with respect to Shapovalov form, and $\sin^{-2}(\pi \alpha(\lambda))$ is an even function of λ, so these terms are symmetric.

For $p = 1, \ldots, n$, the operator $H_p(z)$ is given by the formula

$$H_p(z) = -\sum_\nu h_\nu^{(p)} \partial_{\lambda_\nu} + \pi \sum_{s: s \neq p} \left[\cot(\pi(z_p - z_s)) \Omega^{(p,s)} - \sum_{\alpha \in \Delta_+} \cot(\alpha(\lambda)) (\Omega_\alpha^{(p,s)} - \Omega_{-\alpha}^{(p,s)}) \right].$$

Each ∂_{λ_ν} is symmetric by integration by parts, and $h_\nu^{(p)}$ is symmetric with respect to Shapovalov form. The operator $\Omega^{(p,s)}$ is the symmetric invariant tensor acting on the pth and sth factor, and is symmetric with respect to Shapovalov form. Each $\Omega_\alpha^{(p,s)}$ is adjoint to $\Omega_{-\alpha}^{(p,s)}$ with respect to Shapovalov form. Since $\cot(\alpha(\lambda))$ is an odd function of λ, each $\cot(\alpha(\lambda))(\Omega_\alpha^{(p,s)} - \Omega_{-\alpha}^{(p,s)})$ is self-adjoint with respect to $\langle\, ,\, \rangle$. □

In fact, the elliptic KZB operators are symmetric with respect to $\langle\, ,\, \rangle$ as well.

5.3.1. *Eigenfunctions of* H_0.

Proposition 5.10. *Let* $\beta \in Q$ *be nonzero. For* $\xi \in \mathfrak{h}^*$, $u \in V[0]_{\xi-\rho}$ *and* $v \in V[0]_{\xi+\beta-\rho}$, *we have*

$$\langle \psi_u^\xi, \psi_v^{\xi+\beta} \rangle = 0.$$

Proof. For ξ satisfying (4.2), ψ_u^ξ and $\psi_v^{\xi+\beta}$ have different eigenvalues with respect to H_0, so Proposition 5.9 implies they are orthogonal. The product $\langle \psi_u^\xi, \psi_v^{\xi+\beta} \rangle$ is analytic in ξ, since the functions ψ_u^ξ and $\psi_v^{\xi+\beta}$ are analytic, so the proposition holds for all ξ. □

For generic ξ, Proposition 5.10 implies that the spaces $E(\xi)$ and $E(\xi+\beta)$ are orthogonal for $\beta \in Q$ nonzero. For ξ such that $(\xi, \alpha_j^\vee)$ is an integer for some simple root α_j and certain β, the spaces $E(\xi)$ and $E(\xi+\beta)$ are not disjoint, so are not orthogonal.

Recall that the definition of A_X is $A_X(\mathbf{1}) = \mathbf{1}$ and

$$A_X(F_{\beta_1} \cdots F_{\beta_m}) = \sum_{\sigma \in S_m} A_X^\sigma(F_{\beta_1} \cdots F_{\beta_m})$$

with

$$A_X^\sigma(F_{\beta_1} \cdots F_{\beta_m}) = \prod_{k=1}^m \frac{X_{\beta_{\sigma(k)}}^{a_k^\sigma + 1}}{1 - X_{\beta_{\sigma(1)}} \cdots X_{\beta_{\sigma(k)}}} F_{\beta_1} \cdots F_{\beta_m}$$

where a_k^σ is defined as the cardinality of the subset of $\{k, \dots, m-1\}$ consisting of those j satisfying $\sigma(j) > \sigma(j+1)$. It is clear that $A_X(F_{\beta_1} \cdots F_{\beta_m})$ is zero at $X = 0$ if $m > 0$.

Lemma 5.11. *Let* X^{-1} *denote* $(X_1^{-1}, \dots, X_r^{-1})$. *The map* $A_{X^{-1}} : U(\mathfrak{n}_-) \to \mathcal{A} \otimes U(\mathfrak{n}_-)$ *is regular at* $X = 0$ *with*

$$\lim_{X=0} A_{X^{-1}} = a,$$

where a *is the antipode.*

Proof. We have the formula

$$A_{X^{-1}}^\sigma(F_{\beta_1} \cdots F_{\beta_m}) = \prod_{k=1}^m \frac{X_{\beta_{\sigma(k)}}^{m-k-a_k^\sigma}}{X_{\beta_{\sigma(1)}} \cdots X_{\beta_{\sigma(k)}} - 1} F_{\beta_{\sigma(1)}} \cdots F_{\beta_{\sigma(m)}}.$$

Since a_k^σ equals the cardinality of a subset of $\{k, \dots, m-1\}$, the expression is regular at $X = 0$. In fact, $A_{X^{-1}}^\sigma(F_{\beta_1} \cdots F_{\beta_m})$ is nonzero only if $a_k^\sigma = m - k$ for every k. This holds only if σ is the permutation sending each k to $m - k + 1$. Denoting this permutation by σ_0, we have

$$\lim_{X=0} A_{X^{-1}}^{\sigma_0}(F_{\beta_1} \cdots F_{\beta_m}) = (-1)^m F_{\beta_m} \cdots F_{\beta_2} F_{\beta_1},$$

which is the antipode map a on $U(\mathfrak{n}_-)$. □

Proposition 5.12. [EV2] *For* $\xi \in \mathfrak{h}^*$ *and* $u, v \in V[0]_{\xi-\rho}$, *we have the relation*

$$\langle \psi_u^\xi, \psi_v^\xi \rangle = \langle u, v \rangle_\xi.$$

Proof. Recall that by Proposition 4.6, for $u \in V[0]_{\xi-\rho}$, ψ_u^ξ has the form $\psi_u^\xi(\lambda) = e^{2\pi i \xi(\lambda)} \left(\sum_j A_X(F_j)u_j\right)$ with the vectors u_j defined by the condition $\Xi(\xi-\rho)(\mathbf{1}_{\xi-\rho} \otimes u) = \sum_{j\geqslant 0} F_j \mathbf{1}_{\xi-\rho} \otimes u_j$ for a homogeneous basis $\{F_j\}$ of $U(\mathfrak{n}_-)$. It follows that $\psi_v^\xi(-\lambda)$ has the form

$$\psi_v^\xi(-\lambda) = e^{-2\pi i \xi(\lambda)} \left(\sum_{k\geqslant 0} A_{X^{-1}}(F_k)v_k\right)$$

where $X^{-1} = (X_1^{-1}, \ldots, X_r^{-1})$, and v_k is defined by the condition

$$\Xi(\xi-\rho)(\mathbf{1}_{\xi-\rho} \otimes v) = \sum_{k\geqslant 0} F_k \mathbf{1}_{\xi-\rho} \otimes v_k.$$

Formula (5.3) gives in this case

$$\langle \psi_u^\xi(\lambda), \psi_v^\xi(\lambda)\rangle = (-2\pi i)^{-r} \int_{\tilde{C}} S\left(\sum_j A_X(F_j)u_j, \sum_k A_{X^{-1}}(F_k)v_k\right) \frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_r}{X_r}.$$

The expression $\sum_{j\geqslant 0} A_X(F_j)u_j$ is regular at $X=0$ with value u. Lemma 5.11 gives that $\sum_{k\geqslant 0} A_{X^{-1}}(F_k)v_k$ is regular at $X=0$ with value $\sum_{k\geqslant 0} a(F_k)v_k$, where a denotes the antipode map. The integration is just evaluation at $X=0$:

$$\langle \psi_u^\xi(\lambda), \psi_v^\xi(\lambda)\rangle = S(u, \sum_{k\geqslant 0} a(F_k)v_k).$$

The definition of $\Xi(\xi-\rho)(\mathbf{1}_{\xi-\rho} \otimes v)$ gives each v_k as

$$v_k = \sum_{\ell\geqslant 0} (S_{\xi-\rho}^{-1})_{k\ell}\,\omega(F_\ell)v,$$

so we have

$$\langle \psi_u^\xi(\lambda), \psi_v^\xi(\lambda)\rangle = S\left(u, \sum_{k,\ell\geqslant 0} (S_{\xi-\rho}^{-1})_{k\ell}\, a(F_k)\omega(F_\ell)v\right).$$

The expression $\sum_{k,\ell\geqslant 0} (S_{\xi-\rho}^{-1})_{k\ell}\, a(F_k)\omega(F_\ell)v$ is the definition of $\mathcal{Q}(\xi)v$, so

$$\langle \psi_u^\xi(\lambda), \psi_v^\xi(\lambda)\rangle = S(u, \mathcal{Q}(\xi)v)$$

holds, which gives the proposition. □

6. Bethe ansatz

6.1. **Rational Gaudin Model.** Let $V = V_1 \otimes \cdots \otimes V_n$, where V_p are irreducible highest weight $\mathfrak{g}$-modules of highest weight Λ_p with highest weight vectors v_p. Set $\mathbf{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ and $\Lambda = \sum_p \Lambda_p$. Let $\mathbf{m} = (m_1, \ldots, m_r)$ be a collection of non-negative integers, $m = \sum_j m_j$ and $\mathbf{m}_\alpha = \sum_{j=1}^r m_j \alpha_j$.

The Bethe ansatz gives simultaneous eigenvectors to the operators $K_p(z_1, \ldots, z_n)$ on $\operatorname{Sing} V[\Lambda - \mathbf{m}_\alpha]$.

6.1.1. *Master function.* Let

$$t = (t_1^{(1)}, \dots, t_{m_1}^{(1)}, t_1^{(2)}, \dots, t_{m_2}^{(2)}, \dots, t_1^{(r)}, \dots, t_{m_r}^{(r)}) \in \mathbb{C}^m.$$

We express this ordering of coordinates as $(j,k) < (j',k')$ if either $j < j'$ or $j = j'$ and $k < k'$; here (j,k) corresponds to $t_k^{(j)}$.

The master function $\Phi_K(t, z, \mathbf{\Lambda})$ is defined as

$$\Phi_K(t, z, \mathbf{\Lambda}) = \sum_{(j,k)<(j',k')} (\alpha_j, \alpha_{j'}) \log(t_k^{(j)} - t_{k'}^{(j')}) - \sum_{(j,k)} \sum_{s=1}^{n} (\alpha_j, \Lambda_s) \log(t_k^{(j)} - z_s). \tag{6.1}$$

Critical points of Φ_K with respect to the t variables are are defined as solutions to the equations

$$\sum_{(j',k')\neq(j,k)} \frac{(\alpha_j, \alpha_{j'})}{t_k^{(j)} - t_{k'}^{(j')}} - \sum_{s=1}^{n} \frac{(\alpha_j, \Lambda_s)}{t_k^{(j)} - z_s} = 0, \qquad 1 \leqslant j \leqslant r,\ 1 \leqslant k \leqslant m_i.$$

The group $\Sigma_{\mathbf{m}} = \Sigma_{m_1} \times \cdots \times \Sigma_{m_r}$ acts on the critical set of Φ_K by permutation of coordinates with the same upper index.

6.1.2. *Eigenvectors.* We construct the weight function $u : \mathbb{C}^m \to V[\Lambda - \sum_{j=1}^r m_j \alpha_j]$. Let $\mathbf{b} = (b_1, \dots, b_n)$, with each b_p a non-negative integer and $\sum_{p=1}^n b_p = m$. The collection of these partitions will be denoted B. Let $\Sigma(\mathbf{b})$ denote the set of bijections σ from the set of pairs $\{(p,s) : 1 \leqslant p \leqslant n,\ 1 \leqslant s \leqslant b_p\}$ to the set of variables $\{t_1^{(1)}, \dots, t_{m_1}^{(1)}, \dots, t_1^{(1)}, \dots, t_{m_r}^{(r)}\}$. Let $c(t_k^{(j)}) = j$ be the color function, and set $c_\sigma((p,s)) = c(\sigma((p,s)))$.

For each $\mathbf{b} \in B$ and $\sigma \in \Sigma(\mathbf{b})$, we assign the vector

$$f_{\mathbf{b}}^\sigma v = f_{c_\sigma((1,1))} \cdots f_{c_\sigma((1,b_1))} v_1 \otimes \cdots \otimes f_{c_\sigma((n,1))} \cdots f_{c_\sigma((n,b_n))} v_n.$$

Different σ may give the same $f_{\mathbf{b}}^\sigma$. To $\mathbf{b}$ and σ, we also assign the rational function

$$u_{\mathbf{b}}^\sigma = u_{\mathbf{b},1}^\sigma(z_1) u_{\mathbf{b},2}^\sigma(z_2) \dots u_{\mathbf{b},n}^\sigma(z_n),$$

where

$$u_{\mathbf{b},p}^\sigma(x) = \frac{1}{((\sigma(p,1)) - \sigma(p,2))(\sigma(p,2) - \sigma(p,3)) \cdots (\sigma(p,b_p - 1) - \sigma(p,b_p))(\sigma(p,b_p) - x)}. \tag{6.2}$$

Then we have

$$u(t,z) = \sum_{b \in B} \sum_{\sigma \in \Sigma(\mathbf{b})} u_{\mathbf{b}}^\sigma f_{\mathbf{b}}^\sigma v. \tag{6.3}$$

Theorem 6.1. *Let t_{cr} be an isolated critical point of $\Phi_K(\,\cdot\,, z, \mathbf{\Lambda})$. Then $u(t_{cr}, z)$ is a well defined vector in* $\operatorname{Sing} V[\Lambda - \mathbf{m}_\alpha]$ [MV]*. This vector is an eigenvector of the rational Gaudin operators $K_1(z), \dots, K_n(z)$* [B, BF, RV]*. The eigenvalue of $u(t_{cr}, z)$ with respect to $K_p(z)$ is equal to $\frac{\partial}{\partial z_p} \Phi_K(t_{cr}, z, \mathbf{\Lambda})$* [RV].

Theorem 6.2. [V4] *For an isolated critical point t_{cr} of $\Phi_K(\,\cdot\,, z, \mathbf{\Lambda})$, the vector $u(t_{cr}, z)$ is nonzero.*

For $\mathfrak{g} = sl_{r+1}$ the fact that $u(t_{cr}, z)$ is nonzero is proved in [MTV].

6.1.3. *Norms of eigenvectors.* For t_{cr} a critical point of $\Phi_K(\,\cdot\,, z, \mathbf{\Lambda}, \mathbf{m})$, let

$$\operatorname{Hess}_t \Phi_K(t_{cr}, z, \mathbf{\Lambda}) = \det\left(\frac{\partial^2 \Phi_K}{\partial t_k^{(j)} \partial t_{k'}^{(j')}}\right)(t_{cr})$$

be the Hessian of Φ_K.

Theorem 6.3. [V3] *Let t_{cr} be an isolated critical point of $\Phi_K(\,\cdot\,, z, \mathbf{\Lambda})$. Then*

$$S(u(t_{cr}, z), u(t_{cr}, z)) = \operatorname{Hess}_t \Phi_K(t_{cr}, z, \mathbf{\Lambda})$$

where S is the tensor Shapovalov form on V.

Theorem 6.3 was proved in [MV] for $\mathfrak{g} = sl_{r+1}$.

Theorem 6.4. [V3] *Let t_{cr}, t'_{cr} be isolated critical points of $\Phi_K(\,\cdot\,, z, \mathbf{\Lambda})$ lying in different orbits of $\Sigma_{\mathbf{m}}$. Then*

$$S(u(t_{cr}, z), u(t'_{cr}, z)) = 0.$$

6.2. **Trigonometric Gaudin Model.** Let $V = V_1 \otimes \cdots \otimes V_n$, where V_p are irreducible highest weight $\mathfrak{g}$-modules of highest weight Λ_p with highest weight vectors v_p. Set $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$ and $\Lambda = \sum_p \Lambda_p$, $\mathbf{m} = (m_1, \dots, m_r)$, $m = \sum_j m_j$ and $\mathbf{m}_\alpha = \sum_{j=1}^r m_j \alpha_j$ as above. Then the Bethe ansatz provides simultaneous eigenvectors to the operators $\mathcal{K}_p(z_1, \dots, z_n, \xi)$ in $V[\Lambda - \mathbf{m}_\alpha]$.

6.2.1. *Master function.* In this case we write the master function

$$\Phi_{\mathcal{K}}(t, z, \mathbf{\Lambda}, \mu) = \Phi_K(t, z, \mathbf{\Lambda}) - \sum_{(j,k)} (\alpha_j, \mu) \log(t_k^{(j)}) \tag{6.4}$$

where Φ_K is given by equation (6.1). The function $\Phi_{\mathcal{K}}$ has critical points determined by the equations

$$\sum_{(j',k') \neq (j,k)} \frac{(\alpha_j, \alpha_{j'})}{t_k^{(j)} - t_{k'}^{(j')}} - \sum_{p=1}^n \frac{(\alpha_j, \Lambda_p)}{t_k^{(j)} - z_p} - \frac{(\alpha_j, \mu)}{t_k^{(j)}} = 0 \qquad 1 \leqslant j \leqslant r,\ 1 \leqslant k \leqslant m_j.$$

6.2.2. *Eigenvectors.*

Theorem 6.5. *Let t_{cr} be an isolated critical point of $\Phi_{\mathcal{K}}(\,\cdot\,, z, \mathbf{\Lambda}, \xi - \rho - \frac{1}{2}(\Lambda - \mathbf{m}_\alpha))$. Then $u(t_{cr}, z) \in V[\Lambda - \mathbf{m}_\alpha]$ is an eigenvector of the trigonometric Gaudin operator $\mathcal{K}_p(z_1, \dots, z_n, \xi)$ with eigenvalue equal to*

$$z_p \frac{\partial}{\partial z_p} \Phi_{\mathcal{K}}(t_{cr}, z, \mathbf{\Lambda}, \xi - \rho - \frac{1}{2}(\Lambda - \mathbf{m}_\alpha)) + \frac{1}{2}(\Lambda_p, \Lambda_p + 2\rho)$$

for $p = 1, \dots, n$.

Proof. We relate the construction under question to the Bethe ansatz for the rational Gaudin operators $K_0(0, z_1, \dots, z_n), \dots, K_n(0, z_1, \dots, z_n)$ on the space $\operatorname{Sing} M_\mu \otimes V[\mu + \Lambda - \mathbf{m}_\alpha]$.

We construct the weight function $u_{\mathcal{K}} : \mathbb{C}^m \to M_\mu \otimes V[\mu + \Lambda - \mathbf{m}_\alpha]$. Let $\mathbf{b}_{\mathcal{K}} = (b_0, b_1, \dots, b_n)$, where $\sum_{p=0}^n b_p = m$, and $B_{\mathcal{K}}$ denote the set of these partitions. Let $\Sigma(\mathbf{b}_{\mathcal{K}})$ be the set of bijections σ from $\{(p, s) : 0 \leqslant p \leqslant n,\ 1 \leqslant s \leqslant b_p\}$ to $\{t_1^{(1)}, \dots, t_{m_1}^{(1)}, \dots, t_1^{(1)}, \dots, t_{m_r}^{(r)}\}$. Let $c(t_k^{(j)}) = j$, and $c_\sigma((p, s)) = c(\sigma((p, s)))$.

For each $\mathbf{b}_{\mathcal{K}} \in B_{\mathcal{K}}$ and $\sigma \in \Sigma(\mathbf{b}_{\mathcal{K}})$, we assign the vector

$$f^{\sigma}_{\mathbf{b}_{\mathcal{K}}} v_{\mathcal{K}} = f_{c_\sigma((0,1))} \cdots f_{c_\sigma((0,b_0))} \mathbf{1}_\mu \otimes \cdots \otimes f_{c_\sigma((n,1))} \cdots f_{c_\sigma((n,b_n))} v_n.$$

and the rational function

$$u^{\sigma}_{\mathbf{b}_{\mathcal{K}}} = u^{\sigma}_{\mathbf{b}_{\mathcal{K}},0}(0) u^{\sigma}_{\mathbf{b}_{\mathcal{K}},1}(z_1) \ldots u^{\sigma}_{\mathbf{b}_{\mathcal{K}},n}(z_n),$$

where $u^{\sigma}_{\mathbf{b}_{\mathcal{K}},p}(x)$ is as in equation (6.2). Then

$$u_{\mathcal{K}}(t,z) = \sum_{b_{\mathcal{K}} \in B_{\mathcal{K}}} \sum_{\sigma \in \Sigma(\mathbf{b}_{\mathcal{K}})} u^{\sigma}_{\mathbf{b}_{\mathcal{K}}} f^{\sigma}_{\mathbf{b}_{\mathcal{K}}} v_{\mathcal{K}}.$$

By Theorem 6.1, if t_{cr} is a critical point of $\Phi_{\mathcal{K}}(\,\cdot\,, z, \mathbf{\Lambda}, \mu)$, then $u_{\mathcal{K}}(t_{cr}, z)$ belongs to $\operatorname{Sing} M_\mu \otimes V[\mu + \Lambda - \mathbf{m}_\alpha]$ and is an eigenvector to $K_0(0, z_1, \ldots, z_n)$ with eigenvalue $\sum_{(j,k)} \frac{(\alpha_j,\mu)}{t^{(j)}_k}$ and to $K_p(0, z_1, \ldots, z_n)$ for $p = 1, \ldots, n$ with eigenvalue $\frac{\partial}{\partial z_p} \Phi_{\mathcal{K}}(t_{cr}, z, \mathbf{\Lambda}, \mu)$.

The singular vector has the form

$$u_{\mathcal{K}}(t_{cr}, z) = \mathbf{1}_\mu \otimes u(t_{cr}, z) + \sum_{j>0} F_j \mathbf{1}_\mu \otimes u_j$$

for $u(t_{cr}, z)$ as defined in equation (6.3), since $\mathbf{1}_\mu \otimes u(t_{cr}, z)$ is the sum of the terms of $u_{\mathcal{K}}(t_{cr}, z)$ where $b_0 = 0$. By Corollary 4.3, $u(t_{cr}, z)$ is an eigenfunction of the operators $\mathcal{K}_p\left(z_1, \ldots, z_n, \mu + \rho + \frac{1}{2}(\Lambda - \mathbf{m}_\alpha)\right)$ for $p = 1, \ldots, n$ with eigenvalue $z_p \frac{\partial}{\partial z_p} \Phi_{\mathcal{K}}(t_{cr}, z, \mathbf{\Lambda}, \mu) + \frac{1}{2}(\Lambda_p, \Lambda_p + 2\rho)$. We let $\mu = \xi - \rho - \frac{1}{2}(\Lambda - \mathbf{m}_\alpha)$ for the theorem. □

Proposition 6.6. *Let $\xi - \rho - \frac{1}{2}(\Lambda - \mathbf{m}_\alpha)$ satisfy (2.2). For t_{cr} an isolated critical point of $\Phi_{\mathcal{K}}(\,\cdot\,, z, \mathbf{\Lambda}, \xi - \rho - \frac{1}{2}(\Lambda - \mathbf{m}_\alpha))$, the vector $u(t_{cr}, z)$ is nonzero.*

Proof. By Theorem 6.2, $u_{\mathcal{K}}(t_{cr}, z)$ is nonzero. For μ satisfying (2.2), the singular vector $u_{\mathcal{K}}(t_{cr}, z)$ equals $\Xi(\mu)(\mathbf{1}_\mu \otimes u(t_{cr}, z))$, so $u(t_{cr}, z)$ is nonzero. □

6.2.3. *Norms of eigenvectors.* For t_{cr} a critical point of $\Phi_{\mathcal{K}}(\,\cdot\,, z, \mathbf{\Lambda}, \xi - \rho - \frac{1}{2}(\Lambda - \mathbf{m}_\alpha))$, let $\operatorname{Hess}_t \Phi_{\mathcal{K}}(t_{cr})$ denote the Hessian of $\Phi_{\mathcal{K}}$ with respect to the t variables.

Theorem 6.7. *For t_{cr} an isolated critical point of $\Phi_{\mathcal{K}}(\,\cdot\,, z, \mathbf{\Lambda}, , \xi - \rho - \frac{1}{2}(\Lambda - \mathbf{m}_\alpha))$,*

$$\langle u(t_{cr}, z), u(t_{cr}, z) \rangle_\xi = \operatorname{Hess}_t \Phi_{\mathcal{K}}(t_{cr}, z, \mathbf{\Lambda}, \xi - \rho - \frac{1}{2}(\Lambda - \mathbf{m}_\alpha)).$$

Proof. Let $\mu = \xi - \rho - \frac{1}{2}(\Lambda - \mathbf{m}_\alpha)$. By Theorem 6.3, we have that

$$\operatorname{Hess}_t \Phi_{\mathcal{K}}(t_{cr}, z, \mathbf{\Lambda}, \mu) = S(u_{\mathcal{K}}(t_{cr}, z), u_{\mathcal{K}}(t_{cr}, z))$$

for S the tensor Shapovalov form on $M_\mu \otimes V$. By Corollary 5.4,

$$\langle u(t_{cr}, z), u(t_{cr}, z) \rangle_\xi = S(u_{\mathcal{K}}(t_{cr}, z), u_{\mathcal{K}}(t_{cr}, z)).$$

□

Theorem 6.8. *Let t_{cr}, t'_{cr} be isolated critical points of $\Phi_{\mathcal{K}}(\,\cdot\,, z, \mathbf{\Lambda}, \xi - \rho - \frac{1}{2}(\Lambda - \mathbf{m}_\alpha))$ lying in different orbits of $\Sigma_{\mathbf{m}}$. Then*

$$\langle u(t_{cr}, z), u(t'_{cr}, z) \rangle_\xi = 0$$

Proof. By Corollary 5.4

$$\langle u(t_{cr},z),u(t'_{cr},z)\rangle_\xi = S(u_{\mathcal{K}}(t_{cr},z),u_{\mathcal{K}}(t'_{cr},z)),$$

and by Theorem 6.4 this is zero. □

6.3. **Trigonometric KZB operators.** Let $V = V_1\otimes\cdots\otimes V_n$, where V_p are irreducible highest weight $\mathfrak{g}$-modules of highest weight Λ_p such that V has a non-trivial zero weight subspace $V[0]$. Let v_p denote the highest weight vector of V_p. Set $\boldsymbol{\Lambda} = (\Lambda_1,\dots,\Lambda_n)$ and $\Lambda = \sum_p \Lambda_p$. For $\xi\in\mathfrak{h}^*$, the Bethe ansatz provides simultaneous eigenvectors to the operators $H_p(z_1,\dots,z_n)$ in $E(\xi)$. In this case, $\boldsymbol{m} = (m_1,\dots,m_r)$ is determined by $\boldsymbol{m}_\alpha = \Lambda$, since $E(\xi)$ has values in $V[0]$.

6.3.1. *Master function.* Let $Z_s = e^{-2\pi i z_s}$ and $Z = (Z_1,\dots,Z_n)$. We write the master function

$$\Phi_H(t,z,\boldsymbol{\Lambda},\mu) = \Phi_{\mathcal{K}}(t,Z,\boldsymbol{\Lambda},\mu)$$

with $\Phi_{\mathcal{K}}$ as in (6.4). Critical points of Φ_H with respect to t are determined by the equations

$$\sum_{(j',k')\neq(j,k)} \frac{(\alpha_j,\alpha_{j'})}{t_k^{(j)}-t_{k'}^{(j')}} - \sum_{p=1}^n \frac{(\alpha_j,\Lambda_p)}{t_k^{(j)}-Z_p} - \frac{(\alpha_j,\mu)}{t_k^{(j)}} = 0 \qquad 1\leqslant j\leqslant r,\ 1\leqslant k\leqslant m_j.$$

6.3.2. *Eigenfunctions.*

Theorem 6.9. *Let t_{cr} be an isolated critical point of $\Phi_H(\,\cdot\,,z,\boldsymbol{\Lambda},\xi-\rho)$ Then for $u(t_{cr},Z)\in V[0]$ given by (6.3), $\psi^\xi_{u(t_{cr},Z)}(\lambda)$ is a eigenfunction of H_0 with eigenvalue $\pi i(\xi,\xi)$ and of $H_p(z,\lambda)$ for $p=1,\dots,n$ with eigenvalue $-\frac{1}{2\pi i}\frac{\partial}{\partial z_p}\Phi_H(t_{cr},z,\boldsymbol{\Lambda},\xi-\rho)+(\Lambda_p,\Lambda_p+2\rho)$.*

Proof. By Theorem 6.5, $u(t_{cr},Z)\in V[0]$ is eigenvector of the trigonometric Gaudin operators $\mathcal{K}_p(Z_1,\dots,Z_n,\xi)$ for $p=1,\dots,n$ with eigenvalue $Z_p\frac{\partial}{\partial Z_p}\Phi_{\mathcal{K}}(t_{cr},Z,\boldsymbol{\Lambda},\xi-\rho)+(\Lambda_p,\Lambda_p+2\rho)$. Lemma 4.9 implies that $\psi^\xi_{u(t_{cr},Z)}$ is a eigenfunction of $H_p(z)$ for $p=1,\dots,n$ with the same eigenvalue. □

Proposition 6.10. *For $\xi-\rho\in\mathfrak{h}^*$ satisfying (2.2), and t_{cr} an isolated critical point of $\Phi_H(\,\cdot\,,z,\boldsymbol{\Lambda},\xi-\rho)$, the function $\psi^\xi_{u(t_{cr},Z)}(\lambda)$ is nonzero.*

Proof. By Proposition 6.6, $u(t_{cr},z)$ is nonzero, so $\psi^\xi_{u(t_{cr},Z)}(\lambda)$ is nonzero. □

6.3.3. *Norms of eigenfunctions.* For t_{cr} a critical point of $\Phi_H(\,\cdot\,,z,\boldsymbol{\Lambda},\xi-\rho)$, let $\operatorname{Hess}_t\Phi_H(t_{cr})$ denote the Hessian of Φ_H with respect to the t variables.

Theorem 6.11. *Let t_{cr} be an isolated critical point of $\Phi_H(\,\cdot\,,z,\boldsymbol{\Lambda},\xi-\rho)$. Then*

$$\left\langle \psi^\xi_{u(t_{cr},Z)},\psi^\xi_{u(t_{cr},Z)}\right\rangle = \operatorname{Hess}_t\Phi_H(t_{cr},z,\boldsymbol{\Lambda},\xi-\rho).$$

Proof. By Proposition 5.12,

$$\left\langle \psi^\xi_{u(t_{cr},Z)},\psi^\xi_{u(t_{cr},Z)}\right\rangle = \langle u(t_{cr},Z),u(t_{cr},Z)\rangle_\xi\,.$$

Since t_{cr} is a isolated critical point of $\Phi_{\mathcal{K}}(\,\cdot\,,Z,\boldsymbol{\Lambda},\xi-\rho)$, by Theorem 6.7, we have

$$\langle u(t_{cr},Z),u(t_{cr},Z)\rangle_\xi = \operatorname{Hess}_t\Phi_{\mathcal{K}}(t_{cr},Z,\boldsymbol{\Lambda},\xi-\rho).$$

□

Theorem 6.12. *Let t_{cr}, t'_{cr} be isolated critical points of $\Phi_H(\,\cdot\,, z, \mathbf{\Lambda}, \xi - \rho)$ lying in different $\Sigma_{\mathbf{\Lambda}}$ orbits. Then*

$$\left\langle \psi^{\xi}_{u(t_{cr},Z)}, \psi^{\xi}_{u(t'_{cr},Z)} \right\rangle = 0.$$

Proof. By Proposition 5.12,

$$\left\langle \psi^{\xi}_{u(t_{cr},Z)}, \psi^{\xi}_{u(t'_{cr},Z)} \right\rangle = \langle u(t_{cr}, Z), u(t'_{cr}, Z)\rangle_{\xi}\,,$$

and by Theorem 6.8, this is zero. □

7. Weyl group

For $\alpha \in \Delta$, we have the reflection s_α of $\mathfrak{h}$, defined by $s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha$. The Weyl group W associated to $\mathfrak{g}$ is the group of transformations of $\mathfrak{h}$ generated by such s_α. The simple reflections $s_j = s_{\alpha_j}$ generate W. The Weyl group acts on $V[0]$ so it acts on $V[0]$-valued functions of $\mathfrak{h}$ by $(w\psi)(\lambda) = w(\psi(w^{-1}\lambda))$.

Lemma 7.1. [FW] *The operators $H_0(z, \tau)$, $H_1(z, \tau), \ldots, H_n(z, \tau)$ are Weyl invariant.*

Corollary 7.2. *Let ψ be an eigenfunction of H_p for p any of $0, 1, \ldots, n$. Then for $w \in W$, $w\psi$ is an eigenfunction of H_p with the same eigenvalue.*

7.1. **Scattering matrices.** Following [FV2], we define the maps $T_w(\xi) : V[0] \to V[0]$, rational in the variable $\xi \in \mathfrak{h}^*$ and associated to $w \in W$. For s_j a simple reflection, we define

$$T_{s_j}(\xi) = \sum_{\ell=0}^{\infty} (-1)^\ell \left(\frac{1}{\ell!}\right)^2 \frac{(\xi, \alpha_j^\vee)}{(\xi, \alpha_j^\vee) - \ell} f_j^\ell e_j^\ell. \tag{7.1}$$

For $w \in W$ with decomposition $w = s_{j_m} \cdots s_{j_2} s_{j_1}$ by simple reflections, we define

$$T_w(\xi) = T_{s_{j_m}}(s_{j_{m-1}} \cdots s_{j_2} s_{j_1}\xi) \cdots T_{s_{j_2}}(s_{j_1}\xi) T_{s_{j_1}}(\xi).$$

Proposition 7.3. [FV2] *The map $T_w(\xi)$ does not depend on choice of decomposition $w = s_{j_m} \cdots s_{j_2} s_{j_1}$.*

Corollary 7.4. *For $w_1, w_2 \in W$, the composition property*

$$T_{w_2 w_1}(\xi) = T_{w_2}(w_1\xi) T_{w_1}(\xi).$$

holds.

We note that this map $T_w(\xi)$ is identical to the dynamical Weyl group element $A_w(\xi - \rho)$ acting on $V[0]$, see [TV], [EV1], [STV].

Let $sl_2(j)$ denote the subalgebra of $\mathfrak{g}$ generated by e_j and f_j and let $V_k^{(j)}$ be a $2k+1$-dimensional irreducible $sl_2(j)$ submodule of V.

Proposition 7.5. *For $u \in V_k^{(j)}[0]$, the explicit formula*

$$T_{s_j}(\xi) u = \frac{(1 + \xi_j)(2 + \xi_j) \cdots (k + \xi_j)}{(1 - \xi_j)(2 - \xi_j) \cdots (k - \xi_j)} u$$

holds, where ξ_j denotes $(\xi, \alpha_j^\vee)$.

Proof. Since e_j^{k+1} applied to $u \in V_k^{(j)}[0]$ is zero, we have

$$T_{s_j}(\xi)u = \sum_{\ell=0}^{k} (-1)^\ell \left(\frac{1}{\ell!}\right)^2 \frac{(\xi, \alpha_j^\vee)}{(\xi, \alpha_j^\vee) - \ell} f_j^\ell e_j^\ell u.$$

Each $f_j^\ell e_j^\ell u$ is equal to $\frac{(k+\ell)!}{(k-\ell)!}u$, so we have

$$T_{s_j}(\xi)u = \sum_{\ell=0}^{k} (-1)^\ell \left(\frac{1}{\ell!}\right)^2 \frac{(\xi, \alpha_j^\vee)(k+\ell)!}{((\xi, \alpha_j^\vee) - \ell)(k-\ell)!} u.$$

Thus, $T_{s_j}(\xi)u$ is a degree k rational function of ξ with simple poles at $(\xi, \alpha_j^\vee)$ equal to $1, 2, \dots, k$. The residue at ℓ for $\ell \in \{1, 2, \dots, k\}$ is given by

$$\mathrm{Res}_{(\xi, \alpha_j^\vee) = \ell} T_{s_j}(\xi)u = (-1)^\ell \frac{(k+\ell)!}{\ell!(\ell-1)!(k-\ell)!}.$$

This proves the proposition up to a constant multiple. The constant is fixed by $T_{s_j}(0) = 1$. ☐

Lemma 7.6. [FV2] *Let ξ satisfy (4.2). Then for all $w \in W$, the map $\psi \mapsto w\psi$ is an isomorphism from $E(\xi)$ to $E(w\xi)$.*

Theorem 7.7. [FV2] *The maps $T_w(\xi)$ satisfy the relation*

$$w\psi_u^\xi = \psi_{T_w(\xi)u}^{w\xi}$$

for $u \in V[0]_{\xi - \rho}$.

Let $T_w(\xi)^*$ denote the adjoint operator to $T_w(\xi)$ with respect to the Shapovalov form.

Proposition 7.8. *For a simple reflection s_j, we have that $T_{s_j}(\xi)^* = T_{s_j}(\xi)$.*

Proof. This follows from the fact that $f_j^\ell e_j^\ell$ is self-adjoint with respect to the Shapovalov form. ☐

Corollary 7.9. *For $\xi \in \mathfrak{h}$ and $w = s_{j_m} \cdots s_{j_1}$, we have*

$$T_w(\xi)^* = T_{s_{j_1}}(\xi) T_{s_{j_2}}(s_{j_1}\xi) \cdots T_{s_{j_m}}(s_{j_{m-1}} \cdots s_{j_1}\xi).$$

Lemma 7.10. *For s_j a simple reflection and w_0 the longest element of W, the relation*

$$w_0 T_{s_j}(\xi) w_0^{-1} = T_{w_0 s_j w_0^{-1}}(w_0 s_j \xi).$$

holds.

Proof. We note the relation $w(e_{-\alpha}^l e_\alpha^l)w^{-1} = e_{-w\alpha}^l e_{w\alpha}^l$ [TV]. The simple root $-w_0\alpha_j$ equals $w_0 s_j \alpha_j$, so we have $w_0(f_{\alpha_j}^l e_{\alpha_j}^l)w_0^{-1} = e_{w_0 s_j \alpha_j}^l f_{w_0 s_j \alpha_j}^l$. For $u \in V[0]$, the equation $e_\alpha^l f_\alpha^l u = f_\alpha^l e_\alpha^l u$ holds, so we have

$$w_0 T_{s_j} w_0^{-1} = \sum_{\ell=0}^{\infty} (-1)^\ell \left(\frac{1}{\ell!}\right)^2 \frac{(\xi, \alpha_j^\vee)}{(\xi, \alpha_j^\vee) - \ell} f_{w_0 s_j \alpha_j}^\ell e_{w_0 s_j \alpha_j}^\ell.$$

Since $(\xi, \alpha_j^\vee)$ equals $(w_0 s_j \xi, w_0 s_j \alpha_j^\vee)$ we have

$$w_0 T_{s_j} w_0^{-1} = T_{s_{w_0 s_j \alpha_j}}(w_0 s_j \xi).$$

The simple reflection $s_{w_0 s_j \alpha_j}$ is equal to $w_0 s_j w_0^{-1}$ which gives the lemma. $\square$

Proposition 7.11. *For $w \in W$ and w_0 the longest element of W, the relation*

$$w_0 T_w(\xi)^* w_0^{-1} = T_{w_0 w^{-1} w_0^{-1}}(w_0 w \xi).$$

holds.

Proof. By Corollary 7.9, the operator $T_w(\xi)^*$ can be written as

$$T_w(\xi)^* = T_{s_{i_1}}(\xi) T_{s_{i_2}}(s_{i_1}\xi) \cdots T_{s_{i_k}}(s_{i_{k-1}} \cdots s_{i_1}\xi)$$

for $w = s_{i_k} \cdots s_{i_2} s_{i_1}$. Applying Lemma 7.10 successively to the simple reflections we obtain

$$w_0 T_w(\xi)^* w_0^{-1} = T_{w_0 s_{i_1} w_0^{-1}}(w_0 s_{i_1}\xi) T_{w_0 s_{i_2} w_0^{-1}}(w_0 s_{i_2} s_{i_1}\xi) \cdots T_{w_0 s_{i_k} w_0^{-1}}(w_0 w \xi).$$

The composition property is applied to give the proposition. $\square$

Theorem 7.12. [EV1] *As elements of $End_{\mathbb{C}} V[0] \otimes \mathbb{C}(\mathfrak{h}^*)$, we have*

$$\mathcal{Q}(\xi) = w_0 T_{w_0}(\xi).$$

7.2. **Scalar products.**

Theorem 7.13. *For $u, v \in V[0]_{\xi-\rho}$ and $w \in W$, we have*

$$\langle T_w(\xi)u, T_w(\xi)v \rangle_{w\xi} = \langle u, v \rangle_\xi.$$

Proof. By Theorem 7.12, we have that

$$\langle u, v \rangle_\xi = S(u, w_0 T_{w_0}(\xi) v)$$

and that

$$\langle T_w(\xi)u, T_w(\xi)v \rangle_{w\xi} = S(T_w(\xi)u, w_0 T_{w_0}(w\xi) T_w(\xi) v).$$

The definition of the adjoint operator $T_w(\xi)^*$ gives

$$S(T_w(\xi)u, w_0 T_{w_0}(w\xi) T_w(\xi) v) = S(u, T_w(\xi)^* w_0 T_{w_0}(w\xi) T_w(\xi) v).$$

By Proposition 7.11, we have

$$S(u, T_w(\xi)^* w_0 T_{w_0}(w\xi) T_w(\xi) v) = S(u, w_0 T_{w_0 w^{-1} w_0^{-1}}(w_0 w \xi) T_{w_0}(w\xi) T_w(\xi) v).$$

The cocycle condition gives

$$S(u, w_0 T_{w_0 w^{-1} w_0^{-1}}(w_0 w \xi) T_{w_0}(w\xi) T_w(\xi) v) = S(u, w_0 T_{w_0}(\xi) v)$$

which completes the proof. $\square$

Corollary 7.14. *For $\xi \in \mathfrak{h}^*$, let u and v belong to $V[0]_{\xi-\rho}$. Then*

$$\langle w\psi_u^\xi, w\psi_v^\xi \rangle = \langle \psi_u^\xi, \psi_v^\xi \rangle.$$

holds.

Proof. The left hand side equals $\langle \psi^{w\xi}_{T_w(\xi)u}, \psi^{w\xi}_{T_w(\xi)v} \rangle$ by Theorem 7.7. By Proposition 5.12, the corollary is equivalent to

$$\langle T_w(\xi)u, T_w(\xi)v \rangle_{w\xi} = \langle u, v \rangle_\xi$$

which is the statement of Theorem 7.13 $\square$

Proposition 7.15. *Let $\xi \in \mathfrak{h}^*$ and $w \in W$ be such that $w\xi = \xi + \beta$ for $\beta \in Q$ nonzero. For $u, v \in V[0]_{\xi-\rho}$,*

$$\langle \psi^\xi_u, w\psi^\xi_v \rangle = 0.$$

holds.

Proof. The proposition follows from Proposition 5.10, because of the relation

$$\langle \psi^\xi_u, w\psi^\xi_v \rangle = \langle \psi^\xi_u, \psi^{\xi+\beta}_{T_w(\xi)v} \rangle.$$

□

For $u \in V[0]_{\xi-\rho}$, we define

$$\psi^{W\xi}_u = \sum_{w\in W} (-1)^{l(w)} w\psi^\xi_u$$

where $l(w)$ denotes the length of w. For ξ integral, each $w\xi$ differs from ξ by an element of the root lattice, so $\langle \psi^\xi_u, \psi^{w\xi}_v \rangle$ is well-defined.

Corollary 7.16. *Let $\xi \in \mathfrak{h}^*$ be integral, with $w\xi \neq \xi$ for every $w \in W$. For $u, v \in V[0]_{\xi-\rho}$, the equation*

$$\langle \psi^{W\xi}_u, \psi^{W\xi}_v \rangle = |W| \langle \psi^\xi_u, \psi^\xi_v \rangle.$$

holds.

Proof. By Proposition 7.15, for $w_1 \neq w_2$, $\langle w_1\psi^\xi_u, w_2\psi^\xi_v \rangle$ equals zero. By Corollary 7.14, the terms $\langle w\psi^\xi_u, w\psi^\xi_v \rangle$ for each $w \in W$ are all equal. □

8. Jack polynomials

In this section, we set $\mathfrak{g} = sl_{r+1}$, and let the representation V be $S^{k(r+1)}\mathbb{C}^{r+1}$. Then $V[0]$ is one-dimensional. We set $z_1 = 0$.

Denote the fundamental weights of $\mathfrak{g}$ by $\omega_1, \ldots, \omega_r$. Let P_+ denote the integral dominant weights, the linear combinations of $\omega_1, \ldots, \omega_r$ with non-negative integral coeffecients.

Proposition 8.1. *For $\mu \in \mathfrak{h}$ with $\mu - k\rho \in P_+$, the equality $V[0]_\mu = V[0]$ holds.*

Proof. For

$$\Xi(\mu)(\mathbf{1}_\mu \otimes u) = \sum_{\ell,m\geqslant 0} (S^{-1}_\mu)_{\ell m} F_\ell \mathbf{1}_\mu \otimes \omega(F_m) \otimes u$$

to be defined, it is necessary and sufficient that for each F such that $F\,\mathbf{1}_\mu$ is in the kernel of the Shapovalov form, the expression $\omega(F)u$ is zero. The kernel is generated by the highest weight vectors of the subrepresentations $M_{s_j\cdot\mu}$ of M_μ for simple reflections s_j, which are given by $f_j^{(\mu+\rho,\alpha_j^\vee)}\mathbf{1}_\mu$. Since $(\mu+\rho, \alpha_j^\vee)$ is at least $k+1$, it is sufficient to verify that $e_j^{k+1}u$ is zero for $u \in V[0]$. This is true since the weight spaces $V[(k+1)\alpha_j]$ are empty. □

We use the variables $X_{\omega_j} = e^{-2\pi i\omega_j(\lambda)}$ for ω_j a fundamental weight. For k a fixed non-negative integer, the Jack polynomials $P^{(k)}_\nu$ are a family of Weyl-invariant polynomials in $X^{\pm1}_{\omega_1}, \ldots, X^{\pm1}_{\omega_r}$ parametrized by dominant weights ν. They are characterized by their form

$P_\nu^{(k)} = X_{-\nu} + \sum_{\beta \in Q_+} a_\beta X_{-\nu+\beta}$ for coefficients $a_\beta \in \mathbb{C}$, and their orthogonality with respect to the inner product

$$\langle \phi_1(\lambda), \phi_2(\lambda) \rangle_k = \frac{1}{|W|} \mathrm{const}_{\{X_{\omega_j}\}} \phi_1(\lambda)\phi_2(-\lambda) \prod_{\alpha \in \Delta} (1 - X_\alpha)^{k+1}$$

where $\mathrm{const}_{\{X_{\omega_j}\}}$ is the constant term with respect to the X_{ω_j} variables.

Let Π denote the product $X_{-\rho} \prod_{\alpha \in \Delta_+} (1 - X_\alpha)$.

Proposition 8.2. [FV2] *For $u \in V[0]$ and $\xi \in \mathfrak{h}$ such that $\xi - (k+1)\rho$ belongs to P_+, the antisymmetrized eigenfunction $\psi_u^{W\xi}$ has the form*

$$\psi_u^{W\xi} = \Pi^{k+1} P^{(k)}_{\xi-(k+1)\rho} u.$$

Proposition 8.3. *Let $\xi, \nu \in \mathfrak{h}^*$ be such that $\xi - (k+1)\rho$ and $\nu - (k+1)\rho$ belong to P_+ with $\xi - \nu \in \Delta$. For $u, v \in V[0]$, we have the relation*

$$\langle \psi_u^{W\xi}, \psi_v^{W\nu} \rangle = |W| \langle P^{(k)}_{\xi-(k+1)\rho}, P^{(k)}_{\nu-(k+1)\rho} \rangle_k S(u, v).$$

Proof. The left hand side is defined as

$$\langle \psi_u^{W\xi}, \psi_v^{W\nu} \rangle = \int_C S(\psi_u^{W\xi}(\lambda), \psi_v^{W\nu}(-\lambda)) d\lambda_1 d\lambda_2 \cdots d\lambda_r,$$

where C is given by each $\lambda_j = (\lambda, \alpha_j)$ ranging along the interval from $0 - i\delta$ to $1 - i\delta$. Since $\xi - \nu$ is in Δ, the integrand is a meromorphic function of the variables $X_1, \ldots, X_r$, so the product is

$$\langle \psi_u^{W\xi}, \psi_v^{W\nu} \rangle = \mathrm{const}_{\{X_j\}} S(\psi_u^{W\xi}(\lambda), \psi_v^{W\nu}(-\lambda)).$$

By Proposition 8.2, we have

$$\mathrm{const}_{\{X_j\}} S(\psi_u^{W\xi}, \psi_v^{W\nu}(-\lambda)) = \mathrm{const}_{\{X_j\}} S(\Pi(\lambda)^{k+1} P^{(k)}_{\xi-(k+1)\rho}(\lambda) u, \Pi(-\lambda)^{k+1} P^{(k)}_{\nu-(k+1)\rho}(-\lambda) v).$$

Since $\Pi(-\lambda) = X_\rho \prod_{\alpha \in \Delta_-} (1 - X_\alpha)$, we have

$$\langle \psi_u^{W\xi}, \psi_v^{W\nu} \rangle = \mathrm{const}_{\{X_j\}} \prod_{\alpha \in \Delta} (1 - X_\alpha) P^{(k)}_{\xi-(k+1)\rho}(\lambda) P^{(k)}_{\nu-(k+1)\rho}(-\lambda) S(u, v).$$

On the other hand, we have that

$$|W| \langle P^{(k)}_{\xi-(k+1)\rho}, P^{(k)}_{\nu-(k+1)\rho} \rangle_k S(u, v) = \mathrm{const}_{\{X_{\omega_j}\}} \prod_{\alpha \in \Delta} (1 - X_\alpha) P^{(k)}_{\xi-(k+1)\rho}(\lambda) P^{(k)}_{\nu-(k+1)\rho}(-\lambda).$$

Since the expression on the right has well-defined constant terms in both the X_j variables and the X_{ω_j} variables, and since each X_j is a non-constant product of integer powers of the X_{ω_j} variables, the two constant terms are equal. □

8.1. **Bethe ansatz.** We apply the Bethe ansatz results in this case of $\mathfrak{g} = sl_{r+1}$ and $V = S^{k(r+1)}\mathbb{C}^{r+1}$. The highest weight of V is $k(r+1)\omega_1 = \sum_{j=1}^r k(r+1-j)\alpha_j$. Thus the t variable in $\mathbb{C}^{kr(r+1)/2}$ has variables $t_\ell^{(j)}$ for $1 \leqslant j \leqslant r$ and $1 \leqslant \ell \leqslant k(r+1-j)$. The master function $\Phi_H(t, k(r+1)\omega_1, \xi - \rho)$ and the weight function $u(t)$ are as defined previously, with $z_1 = 1$.

The Bethe ansatz asserts that if t_{cr} is an isolated critical point of $\Phi_H(t, k(r+1)\omega_1, \xi-\rho)$, the function $\psi^\xi_{u(t_{cr})}$ is an eigenfunction of H_0. Theorem 6.11 gives the norm of $\psi^\xi_{u(t_{cr})}$ as

$$\left\langle \psi^\xi_{u(t_{cr})}, \psi^\xi_{u(t_{cr})} \right\rangle = \mathrm{Hess}_t\, \Phi_H(t, k(r+1)\omega_1, \xi-\rho).$$

Proposition 8.4. *If t_{cr} is an isolated critical point of $\Phi_H(t, \nu + k\rho)$, then the relation*

$$\left\langle P^{(k)}_\nu, P^{(k)}_\nu \right\rangle_k S(u(t_{cr}), u(t_{cr})) = \mathrm{Hess}_t\, \Phi_H(t_{cr}, k(r+1)\omega_1, \nu + k\rho).$$

holds.

Proof. By Corollary 7.16, the Bethe function $\psi^{\nu+(k+1)\rho}_{u(t_{cr})}$ has the same norm as its antisymmetrization $\psi^{W(\nu+(k+1)\rho)}_{u(t_{cr})}$ divided by $|W|$, so Proposition 8.3 relates the norm of Jack polynomial to the norm of the Bethe function. □

9. Lie algebra sl_2, one tensor factor

9.1. **Norm of eigenfunction.** We use explicit formulas to show Theorem 6.11 in the case $\mathfrak{g} = sl_2$ and $V = V_\Lambda$ consists of a single irreducible factor with $\Lambda = k\alpha_1$ where k is a non-negative integer. We set $z_1 = 0$, $X = X_1$, $\xi_1 = (\xi, \alpha_1)$ and $\lambda_1 = (\lambda, \alpha_1)$. The space $V[0]$ is one dimensional and we have

$$H_0 = \frac{1}{2\pi i}\frac{d^2}{d\lambda_1^2} + \frac{\pi i k(k+1)}{2\sin^2(\pi\lambda_1)}.$$

Lemma 9.1. *Let $V = V_\Lambda$, with $\Lambda = k\alpha_1$, and $u \in V[0]$. For ξ_1 not equal to any of $\{1, \ldots, k\}$, we have*

$$\left\langle \psi^\xi_u, \psi^\xi_u \right\rangle = \frac{(\xi_1+1)(\xi_1+2)\cdots(\xi_1+k)}{(\xi_1-1)(\xi_1-2)\cdots(\xi_1-k)} S(u,u).$$

Proof. The formula for ψ^ξ_u in this case is calculated in [FV2] as

$$\psi^\xi_u(\lambda) = e^{2\pi i \xi(\lambda)} \sum_{j=0}^k (-1)^j \frac{(k+j)!\Gamma(\xi_1 - j)}{j!(k-j)!\Gamma(\xi_1)} \left(\frac{X}{1-X}\right)^j u.$$

We write

$$\psi^\xi_u(\lambda) = C_{k,\xi}\; e^{2\pi i\xi(\lambda)} P_k^{(-\xi_1,\xi_1)}\left(\frac{1+X}{1-X}\right) u$$

with the Jacobi polynomial [S]

$$P_k^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+k+1)}{k!\Gamma(\alpha+\beta+k+1)} \sum_{j=0}^k \binom{k}{j} \frac{\Gamma(\alpha+\beta+k+j+1)}{\Gamma(\alpha+j+1)} \left(\frac{z-1}{2}\right)^j,$$

and the constant $C_{k,\xi} = \frac{1}{P_k^{(-\xi_1,\xi_1)}(1)}$. The norm of ψ_u^ξ is

$$\begin{aligned}\left\langle \psi_u^\xi, \psi_u^\xi \right\rangle &= \mathrm{const}_X S\left(C_{k,\xi}\ P_k^{(-\xi_1,\xi_1)}\left(\frac{1+X}{1-X}\right)u, C_{k,\xi}\ P_k^{(-\xi_1,\xi_1)}\left(\frac{1+X^{-1}}{1-X^{-1}}\right)u\right)\\ &= C_{k,\xi}^2\ \mathrm{const}_X\ P_k^{(-\xi_1,\xi_1)}\left(\frac{1+X}{1-X}\right)P_k^{(-\xi_1,\xi_1)}\left(\frac{X+1}{X-1}\right)\ S(u,u)\\ &= \frac{P_k^{(-\xi_1,\xi_1)}(-1)}{P_k^{(-\xi_1,\xi_1)}(1)}\ S(u,u).\end{aligned}$$

The special values

$$P_k^{(\alpha,\beta)}(1) = \binom{k+\alpha}{k}$$

and

$$P_k^{(\alpha,\beta)}(-1) = (-1)^k\binom{k+\beta}{k}$$

give

$$\left\langle \psi_u^\xi, \psi_u^\xi \right\rangle = (-1)^k \frac{\Gamma(k+\xi_1+1)\Gamma(-\xi_1+1)}{\Gamma(\xi_1+1)\Gamma(k-\xi_1+1)}$$

which is equivalent to the Lemma. $\square$

We suppress the upper index of the t variables, and write $t = (t_1, \ldots, t_k)$. The master function is

$$\Phi_H(t,z,\Lambda,\xi-\rho) = \sum_{j\neq j'} 2\log(t_j - t_{j'}) - \sum_{j=1}^k 2k\log(t_j-1) - \sum_{j=1}^k(\xi_1-1)\log(t_j).$$

Theorem 9.2. *For t_{cr} a critical point of Φ_H, we have that*

$$\left\langle \psi^\xi_{u(t_{cr},Z)}, \psi^\xi_{u(t_{cr},Z)} \right\rangle = \frac{\left[(\xi_1+1)(\xi_1+2)\cdots(\xi_1+k)\right]^3 k!}{(\xi_1-1)(\xi_1-2)\cdots(\xi_1-k)(k+1)(k+2)\cdots(2k)}.$$

Proof. By applying Lemma 9.1, we need only show

$$S(u(t_{cr},Z), u(t_{cr},Z)) = \left(\frac{(\xi_1+1)(\xi_1+2)\cdots(\xi_1+k)}{(k+1)(k+2)\cdots(2k)}\right)^2 (2k)!.$$

The weight function is

$$u(t,Z) = \sum_{\sigma\in S_k} \frac{1}{(t_{\sigma(1)}-t_{\sigma(2)})(t_{\sigma(2)}-t_{\sigma(3)})\cdots(t_{\sigma(k-1)}-t_{\sigma(k)})(t_{\sigma(k)}-1)} f^k v_\Lambda,$$

which can be simplified to

$$u(t,Z) = \left(\prod_{j=1}^k \frac{1}{t_j-1}\right) f^k v_\Lambda.$$

For t a critical point of Φ_H, [V2] gives the formula

$$\prod_{j=1}^k \frac{1}{t_j-1} = \frac{(\xi_1+1)(\xi_1+2)\cdots(\xi_1+k)}{(k+1)(k+2)\cdots(2k)}.$$

Calculation shows that $S(f^k v_\Lambda, f^k v_\Lambda) = (2k)!$, which gives the theorem. □

By [V2, Equation 1.4.3], we have

$$\mathrm{Hess}_{\mathrm{t}}\left(\Phi_H(t_{cr}, z, \Lambda, \xi-\rho)\right) = k! \prod_{j=0}^{k-1} \frac{(\xi_1+1+j)^3}{(\xi-1-j)(2k-j)}$$

to obtain the claim of Theorem 6.11 that

$$\left\langle \psi^{\xi}_{u(t_{cr},Z)}, \psi^{\xi}_{u(t_{cr},Z)} \right\rangle = \mathrm{Hess}_t\, \Phi_H(t_{cr}, z, \mathbf{\Lambda}, \xi-\rho).$$

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